Complex Analysis

Contents

1 Complex Number System

1.1 Algebraic Structure

Definition 1.1 **Complex Numbers**

The set of complex numbers $\mathbb{C}$ is $\mathbb{R}^2$, together with the operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Theorem 1.1 **Field Structure**

$\mathbb{C}$ is a field. With $0 = (0,0),\ -(x,y) = (-x,-y),\ 1 = (1,0)$ and

$$(x,y)^{-1} = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right) \text{ for } (x,y) \neq (0,0)$$

Notation 1.1

If $z = (x,y)$ we call x the "real part" of z and write $x = \operatorname{Re}(z)$.
We call y the "imaginary part" of z and write $y = \operatorname{Im}(z)$.
Define $i = (0,1)$.

Remark 1.2

There is an embedding

$$\begin{aligned} \mathbb{R} &\hookrightarrow \mathbb{C} \\ x &\longmapsto (x,0) \end{aligned}$$

so we view $\mathbb{R} \subseteq \mathbb{C}$.

Notation 1.3

Under the above embedding, any $z \in \mathbb{C}$ can be written as $z = \operatorname{Re}(z) + \operatorname{Im}(z) \cdot i$
and $i^2 = -1$

1.2 Conjugation and Polar Form

Definition 1.2 **Conjugation**

Define *conjugation* to be the map

$$\begin{aligned} \overline{} : \mathbb{C} &\longrightarrow \mathbb{C} \\ z = x + yi &\longmapsto x - yi = \overline{z} \end{aligned}$$

Note

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2}, \quad \operatorname{Im}(z) = \frac{z - \overline{z}}{2i}$$

Note

If $z = x + yi \in \mathbb{C}$ it has a polar representation $(x, y) \mapsto (r, \theta)$ via $z = r\cos\theta + ir\sin\theta$ where $r = \sqrt{x^2 + y^2} \geq 0, x = r\cos\theta, y = r\sin\theta$. The angle θ is defined modulo 2π when $z \neq 0$, and is arbitrary when $z = 0$.

Remark 1.4

If z_1, z_2 have polar representations

$$z_1 : (r_1, \theta_1), \quad z_2 : (r_2, \theta_2)$$

Then

$$\begin{aligned} z_1 \cdot z_2 &= (r_1\cos\theta_1 + ir_1\sin\theta_1) \cdot (r_2\cos\theta_2 + ir_2\sin\theta_2) \\ &= r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

1.3 Topology and Argument

Note

If $z = x + iy$ and $r = \sqrt{x^2 + y^2}$, then $r^2 = z\overline{z}$.

Remark 1.5

$\mathbb{C}$ is a metric space and thus the triangle inequality holds.

Remark 1.6

$\mathbb{C}$ is a topological space, so topological notions make sense.

Definition 1.3 **Region**

A *region* is a non-empty subset of $\mathbb{C}$ that is open and connected.

Notation 1.7

We write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for the nonzero complex numbers.

Definition 1.4 **Argument**

If $z \in \mathbb{C}^*$ has polar representation (r, θ), we call θ the *argument* of z and write $\theta = \arg(z)$. We'll think about arg as a multivalued function. We write $\operatorname{Arg}(z) = \theta$ if $\theta \in (-\pi, \pi]$, the *principal value*.

Example 1.1

If $z = r(\cos\theta + i\sin\theta)$ and $z \neq 0$ then we can find an n^{th} root of z (actually n of them).

2 Differentiation

2.1 Complex Derivatives

Definition 2.1 **Complex Derivative**

Suppose $f : U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is open and contains z_0. We'll say that f has derivative $f'(z_0)$ at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Theorem 2.1 **Product Rule**

If f and g are differentiable at z_0 then so is fg and

$$(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$

2.2 Cauchy-Riemann Equations

Remark 2.1

If $f : \mathbb{C} \to \mathbb{C}$, then we may write $f(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$. If f is differentiable at $z_0 = x_0 + iy_0$ (say $f'(z_0) = a + ib$), then

$$\begin{aligned} R(z) &= f(z) - f(z_0) - f'(z_0)(z - z_0) \\ &= u(z) - u(z_0) - a(x - x_0) + b(y - y_0) + i(v(z) - v(z_0) - b(x - x_0) - a(y - y_0)) \\ &= R_1(z) + iR_2(z) \end{aligned}$$

Then $\lim_{z \to z_0} R_1 \frac{z}{|z - z_0|} = \lim_{z \to z_0} R_2 \frac{z}{|z - z_0|} = 0$. Thus both u and v are differentiable and $\frac{\partial u}{\partial x}(z_0) = a$, $\frac{\partial u}{\partial y}(z_0) = -b$, $\frac{\partial v}{\partial x}(z_0) = b$, $\frac{\partial v}{\partial y}(z_0) = a$.

Theorem 2.2 **Cauchy-Riemann Equations**

If $f : \mathbb{C} \to \mathbb{C}$, $f(z) = u(z) + iv(z)$, then f is differentiable at z_0 (as a complex function) iff both u and v are differentiable at z_0 and

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \quad \text{and}$$
$$\frac{\partial v}{\partial x}(z_0) = -\frac{\partial u}{\partial y}(z_0)$$

These are called the Cauchy-Riemann equations.

Note

If $u : \mathbb{R}^2 \to \mathbb{R}$, then u is differentiable at (x_0, y_0) if $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0).

2.3 Holomorphic Functions and Exponential

Definition 2.2 **Exponential function**

The exponential function is defined to be the function

$$\exp(x + iy) = e^x(\cos y + i \sin y)$$

we write $e^z := \exp(z)$.

Remark 2.2

For $y \in \mathbb{R}$, we have $e^{iy} = \cos(y) + i\sin(y)$.

Definition 2.3 **Holomorphic**

If $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}$ open, such that f is differentiable at each point in U, then f is *holomorphic*.

Definition 2.4 **Entire**

$f : \mathbb{C} \to \mathbb{C}$ is called *entire* if it is holomorphic on all of $\mathbb{C}$.

2.4 Logarithms and Branches

Definition 2.5 **Logarithm**

A complex number w is a *logarithm* of z if $e^w = z$.

Remark 2.3

The range of the exponential is $\mathbb{C}^*$. So there is no logarithm of 0.

Definition 2.6 **Branch of Logarithm**

If $G \subseteq \mathbb{C}^*$ is a non-empty open set, then a *branch of* log is a continuous function $\ell : G \to \mathbb{C}$ such that for all $z \in G$, $\ell(z)$ is a logarithm of z, i.e. $e^{\ell(z)} = z$

Note

A branch of log may or may not exist depending on the shape of G.

Example 2.1

$G := \mathbb{C} \setminus (-\infty, 0]$. $\ell(z) = \ln|z| + i\,\mathrm{Arg}(z)$. Then

$$\begin{aligned} e^{\ell(z)} &= e^{\ln|z|}(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))) \\ &= |z|(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))) \\ &= z \end{aligned}$$

This is called the *principal branch of* log and is denoted by $\mathrm{Log}(z)$

Note

If we define $\ell(z) = \mathrm{Log}(z) + i2\pi k$ on $\mathbb{C} \setminus (-\infty, 0]$ for fixed $k \in \mathbb{Z}$, then we get a branch of log.

Remark 2.4

It is not possible to find a branch of log on $\mathbb{C}^*$.

2.5 Further Holomorphicity Criteria

Exercise 2.5

In polar form, the Cauchy-Riemann equations become, if $f(z) = u(z) + iv(z)$

$$\begin{aligned} ru_r &= v_\theta \\ \text{and } u_\theta &= -rv_r \end{aligned}$$

Theorem 2.3 **Derivative of Logarithm**

If $\ell : G \to \mathbb{C}$ is a branch of log, then ℓ is holomorphic and $\ell'(z) = \frac{1}{z}$.

Theorem 2.4 **Polynomial Holomorphicity Criterion**

Suppose f is a polynomial in variables x and y (with complex coefficients), then f is holomorphic iff f can be written as polynomial in $z = x + iy$.

3 Series

3.1 Numerical Series

Definition 3.1 **Series**

A *series* is a formal sum $\sum_{n=0}^{\infty} c_n$ where the terms $c_n \in \mathbb{C}$. We say a series *converges* if the sequence of partial sums converge otherwise we say it *diverges*.

Lemma 3.1 **Term Test**

If $\sum_{n=0}^{\infty} c_n$ converges, then $c_n \to 0$.

Definition 3.2 **Absolute Convergence**

A series $\sum_{n=0}^{\infty} c_n$ *converges absolutely* if $\sum_{n=0}^{\infty} |c_n| < \infty$

Lemma 3.2 **Absolute Convergence Test**

If $\sum_{n=0}^{\infty} c_n$ converges absolutely, then $\sum_{n=0}^{\infty} c_n$ converges and $\left|\sum_{n=0}^{\infty} c_n\right| \leq \sum_{n=0}^{\infty} |c_n|$

Example 3.1 **Geometric Series**

$\sum_{n=0}^{\infty} c^n$ converges iff $|c| < 1$. If $|c| < 1$ then $\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$

3.2 Function Series

Definition 3.3 **Convergence of Functions**

Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of $\mathbb{C}$-valued functions on $G \subseteq \mathbb{C}$.

- $(f_n)_{n=1}^{\infty}$ *converges pointwise* to f on $S \subseteq G$ if $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in S$.
- $(f_n)_{n=1}^{\infty}$ *converges uniformly* to f on S if $\sup_{z\in S} |f_n(z) - f(z)| \to 0$
- $(f_n)_{n=1}^{\infty}$ *converges locally uniformly* to f on G if for each $z \in G$, there is some open ball $B \subseteq G$ with centre z such that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on B.

Definition 3.4 **Series of Functions**

If $(f_n)_{n=0}^{\infty}$ is a sequence of functions, $\sum_{n=0}^{\infty} f_n$ is the associated series of functions. The series *converges* to f if the partial sums do.

3.3 Power Series

Notation 3.1

For $z_0 \in \mathbb{C}$ and $r > 0$, write

$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

for the open disk centered at z_0 with radius r, and write

$$\overline{D}(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$$

for the closed disk centered at z_0 with radius r. We write

$$\mathbb{D} = D(0, 1)$$

for the open unit disk.

Example 3.2

$z_0 \in \mathbb{C}$ is fixed. $(f_n)_{n=0}^{\infty}$, where $f_n(z) = a_n(z - z_0)^n$ we get the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. The power series centred at z_0.

Proposition 3.3 **Region of Convergence**

The region of convergence for a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is of the form either

1. $\{z_0\}$
2. $\mathbb{C}$
3. The points in some disk centred at z_0 and possibly points on the boundary.

Example 3.3

$\sum_{n=0}^{\infty} z^n$ has radius of convergence 1. This represents the function $\frac{1}{1-z}$ on $\mathbb{D}$. This converges locally uniformly.

Theorem 3.4 **Comparison Test for Power Series**

Suppose $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z - z_0)^n$ are power series with radius of convergence R_1, R_2 respectively. If $|b_n| \leq M|a_n|$ for some $M > 0$, then $R_1 \leq R_2$

Proof: This follows from the Region of Convergence because if $|z - z_0| < R$, where R is the radius of convergence, then $\sum a_n(z - z_0)^n$ converges absolutely. □

Theorem 3.5 **Cauchy-Hadamard Theorem**

The radius of convergence for a power series $\sum a_n(z - z_0)^n$ is

$$\frac{1}{\limsup_{n \to \infty} |a_n|^{\frac{1}{n}}}$$

Theorem 3.6 **Differentiation of Power Series**

Suppose $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence $R > 0$. Let f be the function it represents on $D(z_0, R)$. Then the radius of convergence for $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ is also R and if g is the function it represents, then f is differentiable and $f'(z) = g(z)$ for $z \in D(z_0, R)$.

Proof: Note

$$\begin{aligned}
&\frac{1}{\limsup_{n\to\infty} |na_n|^{\frac{1}{n}}} \\
&= \frac{1}{\lim n^{\frac{1}{n}}} \cdot \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \\
&= \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \\
&= R
\end{aligned}$$

□

Example 3.4

$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ then f is entire, $f'(z) = f(z)$ and $f(0) = 1$.

3.4 Consequences of Power Series

Corollary 3.7 **Smoothness of Power Series**

If f is represented by a power series on its disk of convergence, then derivatives of all orders exist on that disk.

4 Complex Integration

4.1 Complex Integrals on Intervals

Definition 4.1 **Complex Integral**

Suppose $\varphi : [a, b] \to \mathbb{C}$ is piecewise continuous (continuous except at finitely many points where one-sided limits exist). Then $\operatorname{Re} \varphi$ and $\operatorname{Im} \varphi$ are also piecewise continuous. Set

$$\int_a^b \varphi := \int_a^b \operatorname{Re} \varphi + i \int_a^b \operatorname{Im} \varphi$$

Note this is complex linear in φ

Theorem 4.1 **Fundamental Theorem of Calculus**

If φ is continuous and piecewise $\mathcal{C}^1$ on $[a, b]$, then

$$\int_a^b \varphi'(t)\,\mathrm{d}t = \varphi(b) - \varphi(a)$$

Theorem 4.2 **Integral Triangle Inequality**

$$\left|\int_a^b \varphi(t)\,\mathrm{d}t\right| \leq \int_a^b |\varphi(t)|\,\mathrm{d}t$$

4.2 Path Integrals

Definition 4.2 **Path**

A path is a function $[a, b] \to \mathbb{C}$.

Definition 4.3 **Piecewise $\mathcal{C}^1$ Path**

A path $\gamma : [a, b] \to \mathbb{C}$ is piecewise $\mathcal{C}^1$ if γ is continuous and there is a partition of $[a, b]$ such that γ is $\mathcal{C}^1$ on each subinterval, with one-sided limits for γ' at the breakpoints.

Definition 4.4 **Path Integral**

Suppose $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$ and $f : G \to \mathbb{C}$ is continuous on a set G containing the image of γ. Then define

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

Example 4.1

$f(z) = ie^x$ where $z = x + iy$, $\gamma : [a, b] \to \mathbb{C}$, $\gamma(t) = t$ then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b ie^t\,\mathrm{d}t = i\left(e^b - e^a\right)$$

Theorem 4.3 **Fundamental Theorem for Path Integrals**

If $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$ and if f is holomorphic on an open set that contains the range of γ then

$$\int_\gamma f'(z)\,\mathrm{d}z = f(\gamma(b)) - f(\gamma(a))$$

Proof:

$$\int_\gamma f'(z)\,\mathrm{d}z = \int_a^b f'(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (f\circ\gamma)'(t)\,\mathrm{d}t = (f\circ\gamma)(b) - (f\circ\gamma)(a)$$

□

Proposition 4.4 **Reparametrization Invariance**

If $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$ and $\beta : [a_1, b_1] \to [a, b]$ is increasing and piecewise-$\mathcal{C}^1$ with $\beta(a_1) = a$ and $\beta(b_1) = b$. Set $\gamma_1 = \gamma \circ \beta$. Suppose f is continuous on the image of γ. Then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Proof: By definition of the path integral and the chain rule,

$$\begin{aligned}\int_{\gamma_1} f(z)\,\mathrm{d}z &= \int_{a_1}^{b_1} f(\gamma_1(s))\gamma_{1'}(s)\,\mathrm{d}s \\ &= \int_{a_1}^{b_1} f(\gamma(\beta(s)))\gamma'(\beta(s))\beta'(s)\,\mathrm{d}s\end{aligned}$$

By the substitution theorem with $t = \beta(s)$,

$$\begin{aligned}\int_{a_1}^{b_1} f(\gamma(\beta(s)))\gamma'(\beta(s))\beta'(s)\,\mathrm{d}s &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_\gamma f(z)\,\mathrm{d}z\end{aligned}$$

□

Proposition 4.5 **Properties of Path Integrals**

Suppose f is continuous on the image of the paths below.

1. If $a < c < b$ and $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$, let $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$. Then
$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z.$$
2. If $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$, define the reverse path $-\gamma : [a, b] \to \mathbb{C}$ by $(-\gamma)(t) = \gamma(a + b - t)$. Then
$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

Proof: For the first property, by the definition of path integral,

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^c f(\gamma(t))\gamma'(t)\,\mathrm{d}t + \int_c^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z.\end{aligned}$$

For the second property, by the chain rule,

$$(-\gamma)'(t) = -\gamma'(a+b-t).$$

Hence

$$\int_{-\gamma} f(z)\,\mathrm{d}z = \int_a^b f(\gamma(a+b-t)) \cdot (-\gamma'(a+b-t))\,\mathrm{d}t.$$

Using the substitution $s = a + b - t$, we get

$$\begin{aligned}\int_{-\gamma} f(z)\,\mathrm{d}z &= \int_b^a f(\gamma(s))\gamma'(s)\,\mathrm{d}s \\ &= -\int_a^b f(\gamma(s))\gamma'(s)\,\mathrm{d}s \\ &= -\int_\gamma f(z)\,\mathrm{d}z.\end{aligned}$$

□

Example 4.2

$z_0 \in \mathbb{C}$, $R > 0$, $n \in \mathbb{Z}$, $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + Re^{it}$. If $n \neq -1$, then

$$\int_\gamma (z - z_0)^n \, \mathrm{d}z = 0$$

If $n = -1$, then

$$\int_\gamma (z - z_0)^n \, \mathrm{d}z = 2\pi i$$

Indeed, if $n \neq -1$, then $F(z) = \frac{1}{n+1}(z - z_0)^{n+1}$ is an antiderivative of $(z - z_0)^n$, so

$$\begin{aligned} \int_\gamma (z - z_0)^n \, \mathrm{d}z &= F(\gamma(2\pi)) - F(\gamma(0)) \\ &= \frac{1}{n+1}(\gamma(2\pi) - z_0)^{n+1} - \frac{1}{n+1}(\gamma(0) - z_0)^{n+1} \\ &= 0. \end{aligned}$$

If $n = -1$, then

$$\begin{aligned} \int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z &= \int_0^{2\pi} \frac{1}{\gamma(t) - z_0} \gamma'(t) \, \mathrm{d}t \\ &= \int_0^{2\pi} \frac{1}{Re^{it}} iRe^{it} \, \mathrm{d}t \\ &= \int_0^{2\pi} i \, \mathrm{d}t \\ &= 2\pi i. \end{aligned}$$

4.3 Estimates and Uniform Convergence

Definition 4.5 **Length of a Path**

If $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$ its length is

$$L(\gamma) := \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

Theorem 4.6 **ML Inequality**

If $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$ and f is continuous on the image of γ, then

$$\left| \int_\gamma f(z) \, \mathrm{d}z \right| \leq L(\gamma) M_f$$

where M_f is the maximum value of $|(f \circ \gamma)(t)|$ for $t \in [a, b]$.

Proof:

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\right| \\ &\leq \int_a^b |f(\gamma(t))\gamma'(t)|\,\mathrm{d}t \\ &= \int_a^b |f(\gamma(t))||\gamma'(t)|\,\mathrm{d}t \\ &\leq \int_a^b M_f|\gamma'(t)|\,\mathrm{d}t \\ &= M_f L(\gamma)\end{aligned}$$

□

Theorem 4.7 **Uniform Convergence for Path Integrals**

Suppose $\gamma : [a, b] \to \mathbb{C}$ is piecewise-$\mathcal{C}^1$ and $(f_n)_{n=1}^\infty$ are continuous on the image of γ and (f_n) converges uniformly to f on the range of γ. Then

$$\int_\gamma f = \lim_{n\to\infty} \int_\gamma f_n$$

Proof: Since $f_n \to f$ uniformly and each f_n is continuous on the image of γ, f is continuous on the image of γ. By the ML Inequality,

$$\begin{aligned}\left|\int_\gamma f_n - \int_\gamma f\right| &= \left|\int_\gamma (f_n - f)\right| \\ &\leq L(\gamma) \sup_{t\in[a,b]} |f_n(\gamma(t)) - f(\gamma(t))| \\ &= L(\gamma) \sup_{z\in\gamma([a,b])} |f_n(z) - f(z)|.\end{aligned}$$

Since $f_n \to f$ uniformly on the range of γ, the last expression tends to 0. Hence $\int_\gamma f_n \to \int_\gamma f$. □

4.4 Cauchy's Theorem

Example 4.3 **Fourier Transform of the Gaussian**

Let $a, b > 0$ and let $R_{a,b}$ be the positively oriented boundary of the rectangle with vertices $-a$, a, $a + ib$, and $-a + ib$. Consider

$$\int_{R_{a,b}} e^{-z^2}\,\mathrm{d}z.$$

Since

$$e^{-z^2} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{n!},$$

define

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)n!}.$$

The power series for g has infinite radius of convergence, and termwise differentiation gives $g'(z) = e^{-z^2}$. Hence

$$\int_{R_{a,b}} e^{-z^2} \, \mathrm{d}z = \int_{R_{a,b}} g'(z) \, \mathrm{d}z = 0.$$

Writing the four sides of $R_{a,b}$ separately gives

$$0 = \int_{-a}^{a} e^{-x^2} \, \mathrm{d}x + \int_0^b e^{-(a+iy)^2} i \, \mathrm{d}y - \int_{-a}^{a} e^{-(x+ib)^2} \, \mathrm{d}x - \int_0^b e^{-(-a+iy)^2} i \, \mathrm{d}y.$$

Thus

$$\int_{-a}^{a} e^{-(x+ib)^2} \, \mathrm{d}x = \int_{-a}^{a} e^{-x^2} \, \mathrm{d}x + i \int_0^b e^{-(a+iy)^2} \, \mathrm{d}y - i \int_0^b e^{-(-a+iy)^2} \, \mathrm{d}y.$$

Since

$$e^{-(x+ib)^2} = e^{-x^2+b^2-2ibx},$$

we get

$$\int_{-a}^{a} e^{-x^2} e^{-2ibx} \, \mathrm{d}x = e^{-b^2} \int_{-a}^{a} e^{-x^2} \, \mathrm{d}x + ie^{-b^2} \int_0^b e^{-(a+iy)^2} \, \mathrm{d}y - ie^{-b^2} \int_0^b e^{-(-a+iy)^2} \, \mathrm{d}y.$$

For $0 \le y \le b$,

$$\left| e^{-(a+iy)^2} \right| = e^{-a^2+y^2} \le e^{-a^2+b^2}$$

and the same bound holds for $e^{-(-a+iy)^2}$. Hence the two vertical integrals tend to 0 as $a \to \infty$. Therefore

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-2ibx} \, \mathrm{d}x = e^{-b^2} \int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x.$$

Since $\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}$,

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-2ibx} \, \mathrm{d}x = \sqrt{\pi} e^{-b^2}.$$

Theorem 4.8 **Cauchy's Theorem for a Triangle (Goursat's Lemma)**

If T is the positively oriented boundary of a triangle in $\mathbb{C}$ and if f is holomorphic in a region that contains T and its interior, then

$$\int_T f(z)\,\mathrm{d}z = 0.$$

Proof: Let Δ be the closed filled triangle with boundary T. Let

$$I = \int_T f(z)\,\mathrm{d}z$$

Divide Δ into four congruent closed subtriangles by joining the midpoints of the sides. The boundary integrals over the four subtriangles add to I, because the integrals over the shared sides cancel. Hence one subtriangle, call it Δ_1, with positively oriented boundary $T_1 = \partial\Delta_1$, satisfies

$$\left|\int_{T_1} f(z)\,\mathrm{d}z\right| \geq \frac{|I|}{4}.$$

Repeating this construction gives nested closed filled triangles

$$\Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots$$

with positively oriented boundaries $T_n = \partial\Delta_n$ such that

$$\left|\int_{T_n} f(z)\,\mathrm{d}z\right| \geq \frac{|I|}{4^n}.$$

If P is the perimeter of T and D is the diameter of Δ, then the perimeter of T_n and the diameter of Δ_n are $\frac{P}{2^n}$ and $\frac{D}{2^n}$ respectively.

Since the Δ_n are nested compact sets and their diameters tend to 0, there is a unique point

$$z_0 \in \bigcap_{n=1}^{\infty} \Delta_n.$$

Since f is holomorphic at z_0,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

where $\eta(z) \to 0$ as $z \to z_0$. The first two terms have primitives, so their integrals around T_n are 0. Thus

$$\int_{T_n} f(z)\,\mathrm{d}z = \int_{T_n} \eta(z)(z - z_0)\,\mathrm{d}z.$$

Let $\varepsilon > 0$. Since $\eta(z) \to 0$, for all sufficiently large n we have $|\eta(z)| < \varepsilon$ for all $z \in \Delta_n$. Hence, by the ML Inequality,

$$\left|\int_{T_n} f(z)\,\mathrm{d}z\right| \le \varepsilon \cdot \left(\frac{D}{2^n}\right) \cdot \left(\frac{P}{2^n}\right) = \frac{\varepsilon PD}{4^n}.$$

Combining this with the lower bound gives, for all sufficiently large n,

$$\frac{|I|}{4^n} \le \left|\int_{T_n} f(z)\,\mathrm{d}z\right| \le \frac{\varepsilon PD}{4^n}.$$

Multiplying by 4^n, we get $|I| \le \varepsilon PD$. Since $\varepsilon > 0$ was arbitrary, $I = 0$. □

4.5 Cauchy's Theorem for Convex Regions

Theorem 4.9 **Cauchy's Theorem for Convex Regions**

Suppose G is a convex region and f is holomorphic in G. Then for any piecewise-$\mathcal{C}^1$ closed curve γ in G we have

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Proof: Fix $z_0 \in G$. For $z \in G$, define

$$F(z) = \int_{[z_0,z]} f(w)\,\mathrm{d}w.$$

This is well-defined since G is convex. We claim that $F' = f$.

Let $z \in G$. Since G is open, for h sufficiently small we have $z + h \in G$. By convexity, the triangle with vertices $z_0, z, z + h$ is contained in G, so by Cauchy's Theorem for a Triangle (Goursat's Lemma),

$$\int_{[z_0,z]} f(w)\,\mathrm{d}w + \int_{[z,z+h]} f(w)\,\mathrm{d}w - \int_{[z_0,z+h]} f(w)\,\mathrm{d}w = 0.$$

Hence

$$F(z+h) - F(z) = \int_{[z,z+h]} f(w)\,\mathrm{d}w.$$

Parametrizing the line segment $[z, z + h]$ by $w = z + th$, $0 \le t \le 1$, gives

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f(z+th)\,\mathrm{d}t.$$

Therefore

$$\frac{F(z+h) - F(z)}{h} - f(z) = \int_0^1 (f(z+th) - f(z))\,\mathrm{d}t.$$

Since f is continuous at z, the right side tends to 0 as $h \to 0$. Thus $F'(z) = f(z)$.

Now if $\gamma : [a, b] \to G$ is a piecewise-$\mathcal{C}^1$ closed curve, then by the Fundamental Theorem for Path Integrals,

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) = 0,$$

since $\gamma(a) = \gamma(b)$. □

4.6 Cauchy's Integral Formula

Theorem 4.10 **Cauchy's Formula for the Circle**

Suppose C is a circle oriented counter-clockwise and f is holomorphic in a region that contains C and its interior, then if z is in the interior of C

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

Proof: Let

$$g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \neq z \\ f'(z) & \text{if } \zeta = z. \end{cases}$$

Then g is holomorphic in the interior of C, so by Cauchy's Theorem for Convex Regions,

$$\int_C g(\zeta)\,\mathrm{d}\zeta = 0.$$

Hence

$$\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f(z)\int_C \frac{1}{\zeta - z}\,\mathrm{d}\zeta.$$

It remains to compute the last integral. Write $C(t) = z_0 + Re^{it}$, $0 \le t \le 2\pi$, and set $a = \frac{z - z_0}{R}$. Since z is in the interior of C, $|a| < 1$. Then

$$\begin{aligned}\int_C \frac{1}{\zeta - z}\,\mathrm{d}\zeta &= \int_0^{2\pi} \frac{iRe^{it}}{z_0 + Re^{it} - z}\,\mathrm{d}t \\ &= \int_0^{2\pi} \frac{i}{1 - ae^{-it}}\,\mathrm{d}t \\ &= \int_0^{2\pi} i\sum_{n=0}^{\infty} a^n e^{-int}\,\mathrm{d}t \\ &= 2\pi i.\end{aligned}$$

The third equality uses the geometric series expansion, valid uniformly since $|a| < 1$. The last equality follows by integrating the uniformly convergent series term-by-term: all terms with $n \ge 1$ integrate to 0, and the $n = 0$ term gives $2\pi i$. Therefore

$$\int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = 2\pi i f(z),$$

which gives the result. □

Note

The integral is 0 for z outside the circle

4.7 Consequences of Cauchy's Formula

Corollary 4.11 **Mean Value Property**

Suppose f is holomorphic on a region containing $D(z_0, R)$. If $0 < r < R$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \, dt.$$

Proof: Let C_r be the circle of radius r centered at z_0, oriented counter-clockwise. By Cauchy's Formula for the Circle,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta.$$

Parametrize C_r by $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$. Then $\gamma'(t) = ire^{it}$, so

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it} \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \, dt. \end{aligned}$$

□

Note

This is called the *mean value property* because $f(z_0)$ is equal to the average of the values of f on the circle centered at z_0.

4.8 Cauchy Integrals and Analyticity

Definition 4.6

If φ is a continuous function on the range of some piecewise-C^1 path γ. The *Cauchy Integral* over γ is the function

$$z \longmapsto \int_\gamma \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta$$

This is defined for z not in the range of γ

Definition 4.7 **Distance to a Set**

If $A \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$, define

$$\operatorname{dist}(z_0, A) = \inf_{z \in A} |z - z_0|.$$

Proposition 4.12 **Power Series of a Cauchy Integral**

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise-$\mathcal{C}^1$ path, let $\Gamma = \gamma([a, b])$, and suppose φ is continuous on Γ. Define $f : \mathbb{C} \setminus \Gamma \to \mathbb{C}$ by

$$f(z) = \int_\gamma \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta$$

If $z_0 \notin \Gamma$ and

$$R = \operatorname{dist}(z_0, \Gamma),$$

then f has a power series representation on $D(z_0, R)$.

Proof: Let $z \in D(z_0, R)$. Then for $\zeta \in \Gamma$,

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < 1.$$

Thus

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0 - (z - z_0)} \\ &= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \end{aligned}$$

Therefore

$$f(z) = \int_\gamma \varphi(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \, d\zeta.$$

If $0 < \rho < R$ and $|z - z_0| \leq \rho$, then

$$\left| \frac{z - z_0}{\zeta - z_0} \right| \leq \frac{\rho}{R} < 1$$

for all $\zeta \in \Gamma$. Hence the geometric series converges uniformly for $z \in \overline{D}(z_0, \rho)$ and $\zeta \in \Gamma$, so we may integrate term by term. Thus

$$f(z) = \sum_{n=0}^{\infty} \left(\int_{\gamma} \frac{\varphi(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta \right) (z - z_0)^n.$$

This is a power series centered at z_0. □

Theorem 4.13 **Analyticity of Holomorphic Functions**

Suppose f is holomorphic on a region containing $D(z_0, R)$. Then f has a power series representation centered at z_0 with radius of convergence at least R. More precisely, for $0 < r < R$ and $|z - z_0| < r$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta,$$

and C_r is the positively oriented circle of radius r centered at z_0.

Proof: Fix $0 < r < R$. By Cauchy's Formula for the Circle, for $|z - z_0| < r$,

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

This is a Cauchy integral with $\varphi(\zeta) = \frac{f(\zeta)}{2\pi i}$. By the Power Series of a Cauchy Integral,

$$f(z) = \sum_{n=0}^{\infty} \left(\int_{C_r} \frac{f(\zeta)}{2\pi i (\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta \right) (z - z_0)^n.$$

Hence

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

Since $r < R$ was arbitrary, the radius of convergence is at least R. □

4.9 Derivative Formula

Corollary 4.14 **Cauchy's Integral Formula for Derivatives**

Suppose f is holomorphic on a region containing $D(z_0, R)$. If $0 < r < R$ and C_r is the positively oriented circle of radius r centered at z_0, then for every $n \geq 0$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

Proof: By the Analyticity of Holomorphic Functions, for $|z - z_0| < r$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

Since power series can be differentiated term-by-term inside their radius of convergence,

$$f^{(n)}(z_0) = n! a_n.$$

Substituting the formula for a_n gives

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

□

Corollary 4.15 **Holomorphicity of Derivatives**

If f is holomorphic on a region G, then $f^{(n)}$ is holomorphic on G for every $n \geq 0$.

Proof: Let $z_0 \in G$. Since G is open, there is $R > 0$ such that $D(z_0, R) \subseteq G$. By the Analyticity of Holomorphic Functions, f has a power series representation on this disk. Power series can be differentiated term-by-term inside their radius of convergence, so f' is holomorphic on the disk. Repeating this argument gives that $f^{(n)}$ is holomorphic near z_0 for every $n \geq 0$. Since $z_0 \in G$ was arbitrary, $f^{(n)}$ is holomorphic on G. □

4.10 Operations on Power Series

Proposition 4.16 **Cauchy Product of Power Series**

Suppose $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ has radius of convergence R_1 and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ has radius of convergence R_2. Consider

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n$$

Then the radius of convergence for this power series is at least $\min\{R_1, R_2\}$. If f and g are the functions defined by the first two power series, then for $|z - z_0| < \min\{R_1, R_2\}$,

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n.$$

Proof: Let $R = \min\{R_1, R_2\}$ and define $h(z) = f(z)g(z)$ on $D(z_0, R)$. Since f and g are holomorphic on this disk, so is h. Thus h has a power series representation

$$h(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

on $D(z_0, R)$, where $c_n = h^{(n)} \frac{z_0}{n!}$. By Leibniz's rule,

$$\begin{aligned} h^{(n)}(z_0) &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0) \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot k! a_k \cdot (n-k)! b_{n-k} \\ &= n! \sum_{k=0}^{n} a_k b_{n-k}. \end{aligned}$$

Hence

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Therefore the Cauchy product represents $h = fg$ on $D(z_0, R)$, so its radius of convergence is at least R. □

4.11 Morera's Theorem

Theorem 4.17 **Morera's Theorem**

If G is a region, $f : G \to \mathbb{C}$ is continuous, and $\int_T f = 0$ for every triangle T whose edges and interior are contained in G, then f is holomorphic.

Proof: It is enough to show that f is holomorphic in a neighbourhood of each point. Fix $z_0 \in G$ and choose a disk D centered at z_0 with $D \subseteq G$.

For $z \in D$, define

$$g(z) = \int_{[z_0, z]} f(\zeta) \, \mathrm{d}\zeta.$$

The triangle hypothesis implies that this integral is path independent in D: if $z_1, z_2, z_3 \in D$, then the integral around the triangle with these vertices is 0.

By the same line-segment argument as in the proof of Cauchy's Theorem for Convex Regions, $g' = f$ on D. Therefore g is holomorphic on D. Since derivatives of holomorphic functions are holomorphic by Holomorphicity of Derivatives, $f = g'$ is holomorphic on D. Since z_0 was arbitrary, f is holomorphic on G. □

4.12 Liouville's Theorem and Algebraic Consequences

Theorem 4.18 **Liouville's Theorem**

Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded. Then f is constant.

Proof: Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix $z_0 \in \mathbb{C}$ and $r > 0$. Let C_r be the circle centered at z_0 with radius r, oriented counter-clockwise. By Cauchy's Integral Formula for Derivatives,

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^2} \, \mathrm{d}\zeta.$$

By the ML Inequality,

$$|f'(z_0)| \leq \frac{1}{2\pi} \cdot \left(\frac{M}{r^2}\right) \cdot 2\pi r = \frac{M}{r}.$$

Since this holds for every $r > 0$, letting $r \to \infty$ gives $f'(z_0) = 0$. Since z_0 was arbitrary, $f' = 0$ on $\mathbb{C}$, so f is constant. □

Theorem 4.19 **The Fundamental Theorem of Algebra**

If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a nonconstant polynomial with complex coefficients and $a_n \neq 0$, then there exist $z_1, \cdots, z_n \in \mathbb{C}$ such that

$$p(z) = a_n \prod_{k=1}^{n} (z - z_k).$$

Proof: It is enough to show that p has a zero, since after finding a zero z_1, we may factor $p(z) = (z - z_1)q(z)$ and repeat by induction on the degree.

Suppose, for contradiction, that p has no zero. Then $\frac{1}{p}$ is entire. We claim that $\frac{1}{p}$ is bounded. Since $a_n \neq 0$,

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right).$$

For $|z|$ sufficiently large, the term in parentheses has absolute value at least $\frac{|a_n|}{2}$, so

$$|p(z)| \geq \frac{|a_n|}{2} \cdot |z|^n.$$

Hence $\left|\frac{1}{p(z)}\right| \to 0$ as $|z| \to \infty$, so $\frac{1}{p}$ is bounded outside a large disk. On the closed disk, $\frac{1}{p}$ is continuous, hence bounded. Thus $\frac{1}{p}$ is bounded on $\mathbb{C}$.

By Liouville's Theorem, $\frac{1}{p}$ is constant, so p is constant, contradicting that p is nonconstant. Therefore p has a zero, and the factorization follows by induction. □

5 Zeros and the Identity Theorem

5.1 Orders of Zeros

Definition 5.1 **Zero of Order n**

Suppose f is holomorphic on G and $f(z_0) = 0$. We say z_0 is a *zero of order* n if

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0)$$

but $f^{(n)}(z_0) \neq 0$. A zero of order 1 is called a *simple zero*.

Proposition 5.1 **Infinite-Order Zero**

Suppose G is open, f is holomorphic on G, and f has a zero of infinite order at $z_0 \in G$. Then $f = 0$ on the connected component of G containing z_0.

Proof: Let H be the connected component of G containing z_0. Define

$$A = \{z \in H \mid f \text{ is identically zero on some disk centered at } z\}.$$

Since f has a zero of infinite order at z_0, every coefficient in the power series for f centered at z_0 is 0. Thus f is identically zero on some disk centered at z_0, so $z_0 \in A$ and A is non-empty.

The set A is open in H by definition. We show that A is closed in H. Suppose $z_k \in A$ and $z_k \to z \in H$. For each $m \geq 0$, since f is identically zero near each z_k, we have

$$f^{(m)}(z_k) = 0.$$

Since $f^{(m)}$ is continuous, $f^{(m)}(z) = 0$. Hence all derivatives of f vanish at z, so the power series for f centered at z is identically zero on some disk centered at z. Thus $z \in A$.

Therefore A is non-empty, open, and closed in H. Since H is connected, $A = H$, so $f = 0$ on H. □

Proposition 5.2 **Finite-Order Zeros are Isolated**

Suppose f is holomorphic on an open set G and z_0 is a zero of finite order n. Then there is a disk centered at z_0 on which z_0 is the only zero of f.

Proof: Since z_0 is a zero of order n, the power series for f centered at z_0 has the form

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k$$

with $a_n \neq 0$. Thus

$$f(z) = (z - z_0)^n g(z),$$

where

$$g(z) = \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k.$$

Then g is holomorphic near z_0 and $g(z_0) = a_n \neq 0$. By continuity, $g(z) \neq 0$ on some disk centered at z_0. On this disk, the only zero of $f(z) = (z - z_0)^n g(z)$ is z_0. □

5.2 Identity Theorem

Corollary 5.3 **Identity Theorem**

Suppose f and g are holomorphic on an open set G. If f and g agree on a subset $S \subseteq G$ that has a limit point $z_0 \in G$, then $f = g$ on the connected component of G containing z_0.

Proof: Let $h = f - g$. Then h is holomorphic on G and $h = 0$ on S. Since z_0 is a limit point of S, there is a sequence (z_n) in S with $z_n \to z_0$ and $z_n \neq z_0$. Since h is continuous, $h(z_0) = 0$.

If h had a zero of finite order at z_0, then z_0 would be an isolated zero of h, contradicting that $z_n \to z_0$ with $z_n \neq z_0$ and $h(z_n) = 0$. Hence h has a zero of infinite order at z_0. By the Infinite-Order Zero, $h = 0$ on the connected component of G containing z_0. Thus $f = g$ there. □

6 Convergence and Maximum Principles

6.1 Identity Theorem Applications

Example 6.1

Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. What is $f'\left(\frac{i}{2}\right)$?

Proof: Let $g(z) = z^2$. Then $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for all $n \in \mathbb{N}$, and $\frac{1}{n} \to 0 \in \mathbb{D}$. By the Identity Theorem, $f = g$ on $\mathbb{D}$. Hence

$$f'\left(\frac{i}{2}\right) = g'\left(\frac{i}{2}\right) = 2 \cdot \frac{i}{2} = i.$$

□

6.2 Locally Uniform Limits

Proposition 6.1 **Weierstrass Theorem**

Suppose $G \subseteq \mathbb{C}$ is open and $f_k : G \to \mathbb{C}$ is holomorphic for each $k \in \mathbb{N}$. If $f_k \to f$ locally uniformly on G, then f is holomorphic. Moreover, for each $n \geq 0$, $f_k^{(n)} \to f^{(n)}$ locally uniformly on G.

Proof: Fix $z_0 \in G$ and choose $R > 0$ such that $\overline{D}(z_0, R) \subseteq G$. Let C be the positively oriented circle centered at z_0 with radius R.

If $z \in D(z_0, R)$, then by Cauchy's Formula for the Circle,

$$f_k(z) = \frac{1}{2\pi i} \int_C \frac{f_k(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

Since $f_k \to f$ uniformly on C, Uniform Convergence for Path Integrals gives

$$f(z) = \lim_{k\to\infty} f_k(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Thus f is a Cauchy integral on $D(z_0, R)$, so f is holomorphic there. Since z_0 was arbitrary, f is holomorphic on G.

If $0 < r < R$ and $z \in D(z_0, r)$, then by Cauchy's Integral Formula for Derivatives,

$$f_k^{(n)}(z) - f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f_k(\zeta) - f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta.$$

Since $|\zeta - z| \geq R - r$ for $\zeta \in C$ and $z \in D(z_0, r)$, the ML Inequality gives

$$\left|f_k^{(n)}(z) - f^{(n)}(z)\right| \leq n!\frac{R}{(R-r)^{n+1}}\sup_{\zeta\in C}|f_k(\zeta) - f(\zeta)|.$$

The supremum tends to 0 by local uniform convergence, so $f_k^{(n)} \to f^{(n)}$ uniformly on $D(z_0, r)$. Since z_0 was arbitrary, the convergence is locally uniform on G. □

6.3 Maximum Modulus Principle

Theorem 6.2 **Maximum Modulus Principle**

If G is a region and $f : G \to \mathbb{C}$ is holomorphic and nonconstant, then $|f|$ has no local maximum in G.

Proof: Suppose $|f|$ has a local maximum at $z_0 \in G$. Choose $R > 0$ such that $D(z_0, R) \subseteq G$ and

$$|f(z)| \leq |f(z_0)|$$

for all $z \in D(z_0, R)$.

If $f(z_0) = 0$, then $|f(z)| = 0$ on this disk, so f is locally constant. By the Identity Theorem, f is constant on G, a contradiction. Thus $f(z_0) \neq 0$.

For $0 < r < R$, the Mean Value Property gives

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

Taking absolute values,

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{it})\right|\mathrm{d}t \leq |f(z_0)|.$$

Hence equality holds throughout. Since the continuous function

$$t \longmapsto |f(z_0)| - \left|f(z_0 + re^{it})\right|$$

is nonnegative and has integral 0, it is identically 0. Thus $\left|f(z_0 + re^{it})\right| = |f(z_0)|$ for all t.

Equality in the triangle inequality also forces all values $f(z_0 + re^{it})$ to have the same argument as $f(z_0)$. Therefore

$$f(z_0 + re^{it}) = f(z_0)$$

for all t. Since this holds for every $0 < r < R$, $f = f(z_0)$ on the disk centered at z_0 with radius R. By the Identity Theorem, f is constant on G, contradicting the hypothesis. Therefore $|f|$ has no local maximum in G. □

Corollary 6.3

If G is a region, $f : G \to \mathbb{C}$ is holomorphic and nonconstant, and $K \subseteq G$ is compact and nonempty, then $|f|$ attains its maximum on ∂K.

6.4 Schwarz's Lemma

Theorem 6.4 **Schwarz's Lemma**

Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$, $z_0 \neq 0$, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $f(z) = \lambda z$ for all $z \in \mathbb{D}$.

Proof: Define

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Since $f(0) = 0$, the singularity at 0 is removable, so g is holomorphic on $\mathbb{D}$.

Fix $0 < r < 1$. On the circle $|z| = r$,

$$|g(z)| = \frac{|f(z)|}{|z|} < \frac{1}{r}.$$

By the Maximum Modulus Principle, for $|z| \leq r$ we have $|g(z)| \leq \frac{1}{r}$. Letting $r \to 1$ gives $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Hence, for $z \neq 0$,

$$|f(z)| = |z||g(z)| \leq |z|,$$

and the inequality is also true at $z = 0$.

Now suppose $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. Then $|g(z_0)| = 1$, so $|g|$ has a local maximum in $\mathbb{D}$. By the Maximum Modulus Principle, g is constant. Thus $g(z) = \lambda$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and so $f(z) = \lambda z$ for all $z \in \mathbb{D}$. □

6.5 Minimum Modulus Principle

Proposition 6.5 **Minimum Modulus Principle**

Suppose G is a region and $f : G \to \mathbb{C}$ is holomorphic and nonconstant. If $|f|$ has a local minimum at $z_0 \in G$, then $f(z_0) = 0$.

Proof: Suppose $|f|$ has a local minimum at z_0 and $f(z_0) \neq 0$. Then f is nonzero on some disk centered at z_0, so $\frac{1}{f}$ is holomorphic on that disk. Since $|f|$ has a local minimum at z_0, $\left|\frac{1}{f}\right|$ has a local maximum at

z_0. By the Maximum Modulus Principle, $\frac{1}{f}$ is constant on a disk centered at z_0, so f is constant on that disk. By the Identity Theorem, f is constant on G, a contradiction. Thus $f(z_0) = 0$. □

Example 6.2

The conclusion is possible: if $f(z) = z$ and $K = \overline{\mathbb{D}}$, then $|f|$ attains its minimum at 0, and $f(0) = 0$.

7 Laurent Series

7.1 Examples

Example 7.1

For $z \in \mathbb{D}$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

For $|z| > 1$,

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{\infty} z^{-n-1} = -\sum_{n=1}^{\infty} z^{-n}.$$

Example 7.2

Consider

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

The Laurent expansion depends on the annulus.

If $z \in \mathbb{D}$, then

$$\begin{aligned} \frac{1}{z-2} &= -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \\ -\frac{1}{z-1} &= \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n. \end{aligned}$$

Hence

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n.$$

If $1 < |z| < 2$, then

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}},$$
$$-\frac{1}{z-1} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{\infty} z^{-n-1}.$$

Hence

$$\frac{1}{(z-1)(z-2)} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty} z^{-n}.$$

If $|z| > 2$, then

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} = \sum_{n=0}^{\infty} 2^n z^{-n-1},$$
$$-\frac{1}{z-1} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{\infty} z^{-n-1}.$$

Hence

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} (2^n - 1) z^{-n-1}.$$

7.2 Definition and Convergence

Notation 7.1

If $z_0 \in \mathbb{C}$ and $0 \leq R_1 < R_2 \leq \infty$, write

$$A(z_0; R_1, R_2) = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$$

for the annulus centered at z_0 with inner radius R_1 and outer radius R_2.

Definition 7.1 **Laurent Series**

A *Laurent Series* centered at z_0 is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C}$. This converges at z if both

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

converge. Equivalently, the symmetric partial sums

$$\sum_{n=-N}^{N} a_n (z - z_0)^n$$

converge as $N \to \infty$. In that case,

$$\lim_{N\to\infty} \sum_{n=-N}^{N} a_n (z - z_0)^n$$

is defined to be the sum of the Laurent series.

Theorem 7.1 **Convergence of Laurent Series**

Let

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be a Laurent series. Define

$$R_1 = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}} \quad \text{and} \quad R_2 = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}.$$

Then the positive part converges absolutely and locally uniformly on $D(z_0, R_2)$, and the negative part converges absolutely and locally uniformly where $|z - z_0| > R_1$. Hence, if $R_1 < R_2$, the Laurent series converges absolutely and locally uniformly on the annulus $A(z_0; R_1, R_2)$, and the symmetric partial sums

$$\sum_{n=-N}^{N} a_n (z - z_0)^n$$

converge locally uniformly there. This annulus is called the *annulus of convergence* of the Laurent series.

Proof: By the Cauchy-Hadamard Theorem, the positive part is an ordinary power series with radius of convergence R_2, so it converges absolutely and locally uniformly on $D(z_0, R_2)$.

For the negative part, the root test gives

$$\limsup_{n\to\infty} \left|a_{-n}(z-z_0)^{-n}\right|^{\frac{1}{n}} = \frac{R_1}{|z-z_0|}.$$

Thus the negative part converges absolutely when $\frac{R_1}{|z-z_0|} < 1$, i.e. when $|z-z_0| > R_1$. The same estimate gives locally uniform convergence on compact subsets of $|z-z_0| > R_1$.

On every compact subannulus

$$R_1 < r \leq |z-z_0| \leq R < R_2,$$

both the positive and negative parts converge uniformly. Hence, if $R_1 < R_2$, the Laurent series converges absolutely and the symmetric partial sums converge locally uniformly on $A(z_0; R_1, R_2)$. □

Theorem 7.2 **Differentiating Laurent Series**

Suppose the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

converges on the annulus $A(z_0; R_1, R_2)$. Then f is holomorphic on this annulus, and

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n(z-z_0)^{n-1}.$$

Proof: By the Convergence of Laurent Series, the symmetric partial sums converge locally uniformly on the annulus. Each symmetric partial sum is holomorphic on the annulus, so by the Weierstrass Theorem the limit f is holomorphic.

Again by the Weierstrass Theorem, the derivatives of the symmetric partial sums converge locally uniformly to f'. Since

$$\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=-N}^{N} a_n(z-z_0)^n = \sum_{n=-N}^{N} n a_n(z-z_0)^{n-1},$$

it follows that

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n(z-z_0)^{n-1}.$$

□

7.3 Annuli

Theorem 7.3 **Cauchy's Theorem for an Annulus**

Suppose $0 \leq R_1 < R_2 \leq \infty$ and f is holomorphic on the annulus $A(z_0; R_1, R_2)$. For $R_1 < r < R_2$, let C_r be the positively oriented circle centered at z_0 with radius r. Then $\int_{C_r} f(z)\,\mathrm{d}z$ is independent of r. In other words, if $R_1 < r < s < R_2$, then

$$\int_{C_r} f(z)\,\mathrm{d}z = \int_{C_s} f(z)\,\mathrm{d}z.$$

Proof: Let

$$I(r) = \int_{C_r} f(z)\,\mathrm{d}z.$$

Using the parametrization $z = z_0 + re^{it}$, $0 \leq t \leq 2\pi$, we have

$$I(r) = \int_0^{2\pi} f(z_0 + re^{it}) ire^{it}\,\mathrm{d}t.$$

The integrand has continuous partial derivatives in r and t, so we may differentiate under the integral sign:

$$I'(r) = \int_0^{2\pi} \frac{\partial}{\partial r}(f(z_0 + re^{it}) ire^{it})\,\mathrm{d}t.$$

Now

$$\begin{aligned}\frac{\partial}{\partial r}(f(z_0 + re^{it}) ire^{it}) &= f'(z_0 + re^{it}) ire^{2it} + f(z_0 + re^{it}) ie^{it} \\ &= \frac{\partial}{\partial t}(f(z_0 + re^{it}) e^{it}).\end{aligned}$$

Therefore

$$\begin{aligned}I'(r) &= \int_0^{2\pi} \frac{\partial}{\partial t}(f(z_0 + re^{it}) e^{it})\,\mathrm{d}t \\ &= f(z_0 + re^{2\pi i}) e^{2\pi i} - f(z_0 + r) \\ &= f(z_0 + r) - f(z_0 + r) \\ &= 0.\end{aligned}$$

Hence I is constant on (R_1, R_2), so $\int_{C_r} f(z)\,\mathrm{d}z$ is independent of r. □

7.4 Cauchy's Formula on an Annulus

Theorem 7.4 **Cauchy's Formula for an Annulus**

Suppose $0 \leq R_1 < R_2 \leq \infty$ and f is holomorphic on the annulus $A(z_0; R_1, R_2)$. For $R_1 < r < R_2$, let C_r be the positively oriented circle centered at z_0 with radius r. If

$$R_1 < r_1 < |w - z_0| < r_2 < R_2,$$

then

$$f(w) = \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(z)}{z - w} \, \mathrm{d}z - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(z)}{z - w} \, \mathrm{d}z.$$

Proof: Fix w as above and define

$$g(z) = \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Then g is holomorphic on the annulus. By Cauchy's Theorem for an Annulus,

$$\int_{C_{r_1}} g(z) \, \mathrm{d}z = \int_{C_{r_2}} g(z) \, \mathrm{d}z.$$

Hence

$$\int_{C_{r_2}} \frac{f(z)}{z - w} \, \mathrm{d}z - \int_{C_{r_1}} \frac{f(z)}{z - w} \, \mathrm{d}z = f(w) \int_{C_{r_2}} \frac{1}{z - w} \, \mathrm{d}z - f(w) \int_{C_{r_1}} \frac{1}{z - w} \, \mathrm{d}z.$$

The first integral on the right is $2\pi i$ by Cauchy's Formula for the Circle, since w lies inside C_{r_2}. The second integral is 0 by Cauchy's Theorem for Convex Regions, since w lies outside C_{r_1} and $z \longmapsto \frac{1}{z - w}$ is holomorphic on the disk bounded by C_{r_1}. Therefore

$$\int_{C_{r_2}} \frac{f(z)}{z - w} \, \mathrm{d}z - \int_{C_{r_1}} \frac{f(z)}{z - w} \, \mathrm{d}z = 2\pi i f(w),$$

which gives the formula. □

8 Isolated Singularities

8.1 Laurent Series Representation

Corollary 8.1 **Laurent Series Representation**

Suppose $0 \leq R_1 < R_2 \leq \infty$ and write $A = A(z_0; R_1, R_2)$. If f is holomorphic on A, then f has a Laurent series representation on A:

$$f(w) = \sum_{n=-\infty}^{\infty} a_n (w - z_0)^n.$$

If $R_1 < r < R_2$ and C_r is the positively oriented circle centered at z_0 with radius r, then

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

for every $n \in \mathbb{Z}$.

Proof: Fix $w \in A$, and choose r_1, r_2 such that

$$R_1 < r_1 < |w - z_0| < r_2 < R_2.$$

By Cauchy's Formula for an Annulus,

$$f(w) = \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(z)}{z - w} \, \mathrm{d}z - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(z)}{z - w} \, \mathrm{d}z.$$

On C_{r_2} we have $|w - z_0| < |z - z_0|$, so

$$\frac{1}{z - w} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{w - z_0}{z - z_0}} = \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^{n+1}},$$

locally uniformly in w. Hence the outer integral contributes

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \right) (w - z_0)^n.$$

On C_{r_1} we have $|z - z_0| < |w - z_0|$, so

$$\frac{1}{z - w} = -\frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} = -\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}},$$

locally uniformly in w. Because the inner integral is subtracted, this contributes

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_{r_1}} f(z)(z - z_0)^n \, \mathrm{d}z \right) (w - z_0)^{-n-1}.$$

These are exactly the Laurent coefficients above. The coefficient formula is independent of the chosen radius r by Cauchy's Theorem for an Annulus applied to $z \longmapsto \frac{f(z)}{(z - z_0)^{n+1}}$. □

8.2 Classification

Notation 8.1

For $z_0 \in \mathbb{C}$ and $r > 0$, write

$$D^*(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$$

for the punctured disk centered at z_0 with radius r.

Definition 8.1 — **Isolated Singularity**

Suppose G is a region and $f : G \to \mathbb{C}$ is holomorphic. A point z_0 is an *isolated singularity of f* if $z_0 \notin G$ and there is some $R > 0$ such that

$$D^*(z_0, R) \subseteq G.$$

Note

If z_0 is an isolated singularity of f, then for sufficiently small $R > 0$, f is holomorphic on the punctured disk $D^*(z_0, R)$. Thus f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

near z_0.

Definition 8.2 — **Removable Singularity**

An isolated singularity z_0 of $f : G \to \mathbb{C}$ is *removable* if there is a holomorphic function $F : G \cup \{z_0\} \to \mathbb{C}$ such that $F|_G = f$.

Proposition 8.2 — **Removable Singularity Criterion**

Let z_0 be an isolated singularity of f, and write the Laurent series of f near z_0 as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Then z_0 is removable if and only if $a_n = 0$ for all $n < 0$.

Proof: If z_0 is removable, then the holomorphic extension has a power series at z_0, so the Laurent series has no negative-power terms. Conversely, if all negative-power terms vanish, then the Laurent series is a power series in a disk centered at z_0, so defining $F(z_0) = a_0$ gives a holomorphic extension. □

Example 8.1

The function $f(z) = z$ on $\mathbb{C}^*$ has a removable singularity at 0.

Example 8.2

The function

$$f(z) = \frac{z}{e^z - 1}$$

on $\mathbb{C}^*$ has a removable singularity at 0, since $e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$.

Definition 8.3 **Pole**

An isolated singularity z_0 of f is a *pole of order* m if, in the Laurent series of f near z_0,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$$

with $a_{-m} \neq 0$. A pole of order 1 is called a *simple pole*.

Proposition 8.3 **Characterization of Poles**

Let z_0 be an isolated singularity of f. Then f has a pole of order m at z_0 if and only if $(z - z_0)^m f(z)$ has a removable singularity at z_0 and its removable extension is nonzero at z_0.

Proof: If f has a pole of order m, then

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$$

with $a_{-m} \neq 0$. Hence

$$(z - z_0)^m f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^{n+m} = \sum_{k=0}^{\infty} a_{k-m} (z - z_0)^k,$$

so $(z - z_0)^m f(z)$ has a removable singularity at z_0, and the value of its extension at z_0 is $a_{-m} \neq 0$.

Conversely, if $(z - z_0)^m f(z)$ has a removable singularity at z_0 with nonzero value, write its holomorphic extension as

$$g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

with $b_0 \neq 0$. Then

$$f(z) = (z - z_0)^{-m} g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^{k-m},$$

so the first nonzero term is the $(z - z_0)^{-m}$ term. Thus f has a pole of order m at z_0. □

Example 8.3

The function $f(z) = \frac{1}{z}$ has a simple pole at 0.

Definition 8.4 **Essential Singularity**

An isolated singularity z_0 of f is an *essential singularity* if the Laurent series of f near z_0 has infinitely many nonzero negative-power terms.

Proposition 8.4 **Classification of Isolated Singularities**

Every isolated singularity is exactly one of the following: removable, a pole, or essential.

Proof: Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be the Laurent series near the isolated singularity z_0. If $a_n = 0$ for all $n < 0$, then z_0 is removable. If there are finitely many nonzero negative-power terms, let $-m$ be the smallest index with $a_{-m} \neq 0$. Then z_0 is a pole of order m. If there are infinitely many nonzero negative-power terms, then z_0 is essential. □

Example 8.4

The function

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

has an essential singularity at 0.

8.3 Singularity at Infinity

Definition 8.5 **Isolated Singularity at Infinity**

Suppose G is a region and $f : G \to \mathbb{C}$ is holomorphic. The point ∞ is an isolated singularity of f if there exists some $R > 0$ such that

$$\{z \in \mathbb{C} \mid |z| > R\} \subseteq G.$$

Definition 8.6 **Classification at Infinity**

Suppose ∞ is an isolated singularity of f. Define

$$g(w) = f\left(\frac{1}{w}\right)$$

for $0 < |w| < \frac{1}{R}$. We say that ∞ is a removable singularity, a pole of order m, or an essential singularity of f if 0 is respectively a removable singularity, a pole of order m, or an essential singularity of g.

Example 8.5

The function $f(z) = \frac{1}{z}$ has a removable singularity at ∞, since $f\left(\frac{1}{w}\right) = w$ has a removable singularity at 0.

8.4 Removable Singularities

Theorem 8.5 **Removable Singularity Theorem**

Let z_0 be an isolated singularity of $f : G \to \mathbb{C}$. Then z_0 is removable if and only if f is bounded on some punctured disk around z_0. That is, there exist $r > 0$ and $M > 0$ such that

$$D^*(z_0, r) \subseteq G$$

and $|f(z)| \leq M$ for all $z \in D^*(z_0, r)$.

Proof: First suppose z_0 is removable. Let $F : G \cup \{z_0\} \to \mathbb{C}$ be a holomorphic extension of f. Since F is continuous at z_0, there is some $r > 0$ such that F is bounded on

$$D(z_0, r).$$

Hence f is bounded on the punctured disk $D^*(z_0, r)$.

Conversely, suppose $|f(z)| \leq M$ for all $z \in D^*(z_0, r)$. Write the Laurent series of f near z_0 as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We show that all negative coefficients vanish. Fix $n \geq 1$. For $0 < \rho < r$, let C_ρ be the positively oriented circle centered at z_0 with radius ρ. By the Laurent Series Representation,

$$a_{-n} = \frac{1}{2\pi i} \int_{C_\rho} f(z)(z - z_0)^{n-1} \, \mathrm{d}z.$$

Therefore, by the ML Inequality,

$$|a_{-n}| \leq \frac{1}{2\pi} \cdot M\rho^{n-1} \cdot 2\pi\rho = M\rho^n.$$

Since this holds for every $0 < \rho < r$, letting $\rho \to 0$ gives $a_{-n} = 0$. Thus $a_k = 0$ for all $k < 0$, so by the Removable Singularity Criterion, z_0 is removable. □

Theorem 8.6 **Pole Characterization by Growth**

Suppose z_0 is an isolated singularity of f. Then z_0 is a pole if and only if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Proof: First suppose z_0 is a pole of order m. By the Characterization of Poles, the function

$$g(z) = (z - z_0)^m f(z)$$

has a removable singularity at z_0, and its removable extension satisfies $g(z_0) \neq 0$. Hence there are $r > 0$ and $c > 0$ such that $|g(z)| \geq c$ for all $z \in D(z_0, r)$. Thus, for $z \in D^*(z_0, r)$,

$$|f(z)| = \frac{|g(z)|}{|z - z_0|^m} \geq \frac{c}{|z - z_0|^m}.$$

Letting $z \to z_0$ gives $|f(z)| \to \infty$.

Conversely, suppose

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Then there is $r > 0$ such that $f(z) \neq 0$ for all $z \in D^*(z_0, r)$. Define $h(z) = \frac{1}{f(z)}$ on this punctured disk. Since $|h(z)| \to 0$ as $z \to z_0$, the function h is bounded near z_0, so by the Removable Singularity Theorem, h has a removable singularity at z_0. Define its extension by $h(z_0) = 0$.

The zero of h at z_0 has finite order. Indeed, h is not identically zero on any punctured disk around z_0, since f is finite there, so h cannot have a zero of infinite order at z_0. Thus

$$h(z) = (z - z_0)^m k(z)$$

for some $m \geq 1$, where k is holomorphic near z_0 and $k(z_0) \neq 0$. Therefore

$$f(z) = \frac{1}{h(z)} = (z - z_0)^{-m} \frac{1}{k(z)},$$

where $\frac{1}{k}$ is holomorphic near z_0. By the Characterization of Poles, z_0 is a pole of order m. □

Theorem 8.7 **Picard's Theorem**

Let z_0 be an essential singularity of $f : G \to \mathbb{C}$. For every $\varepsilon > 0$ such that

$$D^*(z_0, \varepsilon) \subseteq G,$$

the image

$$f(D^*(z_0, \varepsilon))$$

omits at most one complex number.

Note

The proof of Picard's theorem is out of scope.

Theorem 8.8 **Casorati-Weierstrass Theorem**

Let z_0 be an essential singularity of $f : G \to \mathbb{C}$. For every $\varepsilon > 0$ such that

$$D^*(z_0, \varepsilon) \subseteq G,$$

the image

$$f(D^*(z_0, \varepsilon))$$

is dense in $\mathbb{C}$.

Proof: Suppose not. Then there exist $\varepsilon > 0$, $w \in \mathbb{C}$, and $\delta > 0$ such that

$$|f(z) - w| \geq \delta$$

for all $z \in D^*(z_0, \varepsilon)$. In particular, $f(z) \neq w$ on this punctured disk, so

$$g(z) = \frac{1}{f(z) - w}$$

is holomorphic there. Moreover, $|g(z)| \leq \frac{1}{\delta}$, so by the Removable Singularity Theorem, g has a removable singularity at z_0.

Let G_0 be the holomorphic extension of g to a disk centered at z_0. If $G_0(z_0) \neq 0$, then $\frac{1}{G_0}$ is holomorphic in a smaller disk centered at z_0, so

$$f(z) = w + \frac{1}{G_0}(z)$$

has a removable singularity at z_0, a contradiction.

If $G_0(z_0) = 0$, then $|g(z)| \to 0$ as $z \to z_0$, so

$$|f(z) - w| = \frac{1}{|g(z)|} \to \infty.$$

By Pole Characterization by Growth, $f - w$ has a pole at z_0, and hence f has a pole at z_0, again a contradiction. Therefore the image of every punctured disk around z_0 is dense in $\mathbb{C}$. □

9 Cauchy's Theorem

9.1 Continuous Logarithms Along Curves

Note

Recall Cauchy's Theorem for Convex Regions: if G is a convex region and f is holomorphic on G, then $\int_\gamma f(z)\, \mathrm{d}z = 0$ for every piecewise-$\mathcal{C}^1$ closed curve γ in G. We wish to generalize this theorem to *simply connected* regions.

Theorem 9.1 **Path Lifting for the Exponential**

Suppose $a < b$ and $\varphi : [a, b] \to \mathbb{C}^*$ is continuous. Then there exists a continuous function $\psi : [a, b] \to \mathbb{C}$ such that

$$e^{\psi(t)} = \varphi(t)$$

for all $t \in [a, b]$. Moreover, ψ is unique up to adding a constant in $2\pi i\mathbb{Z}$.

Proof: Since φ is continuous and $[a, b]$ is compact, there is $m > 0$ such that

$$|\varphi(t)| \geq m$$

for all $t \in [a, b]$. By uniform continuity, choose a partition

$$a = t_0 < t_1 < \ldots < t_N = b$$

such that if $s, t \in [t_j, t_{j+1}]$, then

$$|\varphi(t) - \varphi(s)| < \frac{m}{2}.$$

Hence, for $t \in [t_j, t_{j+1}]$,

$$\left|\frac{\varphi(t)}{\varphi(t_j)} - 1\right| = \frac{|\varphi(t) - \varphi(t_j)|}{|\varphi(t_j)|} < \frac{\frac{m}{2}}{m} = \frac{1}{2}.$$

In particular, $\frac{\varphi(t)}{\varphi(t_j)}$ lies in the disk $D\left(1, \frac{1}{2}\right)$, so the principal logarithm is defined there.

Choose $c_0 \in \mathbb{C}$ such that $e^{c_0} = \varphi(t_0)$. Inductively, once c_j is chosen with $e^{c_j} = \varphi(t_j)$, define

$$\psi(t) = c_j + \operatorname{Log}\left(\frac{\varphi(t)}{\varphi(t_j)}\right)$$

for $t \in [t_j, t_{j+1}]$, and set $c_{j+1} = \psi(t_{j+1})$. The next piece agrees at t_{j+1} because

$$c_{j+1} + \operatorname{Log}\left(\frac{\varphi(t_{j+1})}{\varphi(t_{j+1})}\right) = c_{j+1} + \operatorname{Log}(1) = c_{j+1}.$$

Thus the definitions agree at the endpoints. Moreover,

$$e^{\psi(t)} = e^{c_j} \frac{\varphi(t)}{\varphi(t_j)} = \varphi(t).$$

Since each piece is continuous, ψ is continuous on all of $[a, b]$.

For uniqueness, suppose ψ_1 and ψ_2 are two such functions. Then

$$e^{\psi_1(t) - \psi_2(t)} = 1$$

for all t, so

$$\frac{\psi_1(t) - \psi_2(t)}{2\pi i} \in \mathbb{Z}.$$

This is a continuous integer-valued function on the connected interval $[a, b]$, so it is constant. Hence $\psi_1 - \psi_2 \in 2\pi i\mathbb{Z}$ is constant. □

Note

The map

$$z \longmapsto e^z : \mathbb{C} \to \mathbb{C}^*$$

is a covering map, and the theorem says that every path φ in $\mathbb{C}^*$ can be lifted to a path ψ in $\mathbb{C}$ such that $e^\psi = \varphi$. The ambiguity by constants in $2\pi i\mathbb{Z}$ reflects the fact that the points over a fixed $z \in \mathbb{C}^*$ differ by multiples of $2\pi i$.

Corollary 9.2 **Continuous Logarithm Along a Curve**

Suppose $\gamma : [a, b] \to \mathbb{C}$ is continuous and f is continuous on the range of γ. If $f(\gamma(t)) \neq 0$ for all $t \in [a, b]$, then there exists a continuous function $\psi : [a, b] \to \mathbb{C}$ such that

$$e^{\psi(t)} = (f \circ \gamma)(t)$$

for all $t \in [a, b]$.

Proof: Apply Path Lifting for the Exponential to the continuous function

$$\varphi = f \circ \gamma : [a, b] \to \mathbb{C}^*.$$

□

9.2 Winding Number

Definition 9.1 **Increment of Logarithm**

Suppose $\gamma : [a, b] \to \mathbb{C}$ is continuous and f is continuous on $\gamma([a, b])$ with $f(\gamma(t)) \neq 0$ for all $t \in [a, b]$. Let $\psi : [a, b] \to \mathbb{C}$ be a continuous logarithm of $f \circ \gamma$, so

$$e^{\psi(t)} = f(\gamma(t)).$$

The *increment of* $\log f$ *along* γ is

$$\Delta(\log f, \gamma) = \psi(b) - \psi(a).$$

Note

This is well-defined: by Path Lifting for the Exponential, any two choices of ψ differ by a constant in $2\pi i\mathbb{Z}$, so the difference $\psi(b) - \psi(a)$ is independent of the choice.

Example 9.1

Let $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$, and let $f(z) = z$. Then $\psi(t) = it$ is a continuous logarithm of $f \circ \gamma$, so

$$\Delta(\log f, \gamma) = 2\pi i.$$

Definition 9.2 **Increment of Argument**

Under the same hypotheses, the *increment of* $\arg f$ *along* γ is

$$\Delta(\arg f, \gamma) := \operatorname{Im}(\Delta(\log f, \gamma)).$$

Note

This matches the usual idea of change in argument. If ψ is a continuous logarithm of $f \circ \gamma$, write $\psi(t) = u(t) + i\theta(t)$. Since

$$f(\gamma(t)) = e^{u(t)}e^{i\theta(t)},$$

we have $u(t) = \ln|f(\gamma(t))|$ and $\theta(t)$ is a continuous choice of argument for $f(\gamma(t))$. Thus

$$\psi(t) = \ln|f(\gamma(t))| + i\theta(t),$$

so taking the imaginary part of the change in ψ records exactly the change in the chosen argument:

$$\Delta(\arg f, \gamma) = \theta(b) - \theta(a).$$

Note

Equivalently, we can first normalize the curve $f \circ \gamma$ to the unit circle by defining

$$h(t) = \frac{f(\gamma(t))}{|f(\gamma(t))|}.$$

Then $h : [a, b] \to S^1$. A continuous choice of argument is the same as a lift $\theta : [a, b] \to \mathbb{R}$ through the map $\theta \mapsto e^{i\theta}$, meaning $h(t) = e^{i\theta(t)}$. With this viewpoint,

$$\Delta(\arg f, \gamma) = \theta(b) - \theta(a).$$

Definition 9.3 **Winding Number**

Suppose $\gamma : [a, b] \to \mathbb{C}$ is a continuous closed curve and $z_0 \notin \gamma([a, b])$. The *winding number* of γ around z_0 is

$$\operatorname{Ind}_\gamma(z_0) := \frac{1}{2\pi}\Delta(\arg(z - z_0), \gamma) \in \mathbb{Z}.$$

Note

Since γ is closed, if ψ is a continuous logarithm of $\gamma(t) - z_0$, then $e^{\psi(a)} = e^{\psi(b)}$. Hence $\psi(b) - \psi(a) \in 2\pi i\mathbb{Z}$, so $\operatorname{Ind}_\gamma(z_0) \in \mathbb{Z}$.

Example 9.2

If $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$, then $\operatorname{Ind}_\gamma(0) = 1$.

Proposition 9.3 **Integral Formula for Winding Number**

Suppose $\gamma : [a, b] \to \mathbb{C}$ is a piecewise-$\mathcal{C}^1$ closed curve and $z_0 \notin \gamma([a, b])$. Then

$$\operatorname{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, \mathrm{d}t = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, \mathrm{d}z.$$

Proof: By Continuous Logarithm Along a Curve, there is a continuous function $\psi : [a, b] \to \mathbb{C}$ such that

$$e^{\psi(t)} = \gamma(t) - z_0.$$

On each interval where γ is $\mathcal{C}^1$, the function $\gamma(t) - z_0$ is $\mathcal{C}^1$ and nonzero. Locally, ψ differs from a branch of $\log(\gamma(t) - z_0)$ by a constant in $2\pi i\mathbb{Z}$, so ψ is piecewise-$\mathcal{C}^1$. Differentiating gives

$$e^{\psi(t)}\psi'(t) = \gamma'(t),$$

so

$$\psi'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}.$$

Therefore

$$\Delta(\log(z - z_0), \gamma) = \psi(b) - \psi(a) = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0}\, \mathrm{d}t.$$

Since γ is closed, this increment lies in $2\pi i\mathbb{Z}$, and

$$\Delta(\log(z - z_0), \gamma) = i\Delta(\arg(z - z_0), \gamma).$$

Dividing by $2\pi i$ gives the first equality. The second equality is the definition of the path integral. □

Proposition 9.4 **Continuity of Winding Number**

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise-$\mathcal{C}^1$ closed curve, then

$$\begin{aligned} \operatorname{Ind}_\gamma : \mathbb{C} \setminus \gamma([a, b]) &\longrightarrow \mathbb{Z} \\ z_0 &\longmapsto \operatorname{Ind}_\gamma(z_0) \end{aligned}$$

is continuous.

Proof: Let $z_0 \notin \gamma([a, b])$. Since $\gamma([a, b])$ is compact, there is $d > 0$ such that

$$|\gamma(t) - z_0| \geq d$$

for all $t \in [a, b]$. If $|w - z_0| < \frac{d}{2}$, then $w \notin \gamma([a, b])$ and

$$|\gamma(t) - w| \geq \frac{d}{2}$$

for all t. By Integral Formula for Winding Number,

$$\operatorname{Ind}_\gamma(w) - \operatorname{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_a^b \gamma'(t) \left(\frac{1}{\gamma(t) - w} - \frac{1}{\gamma(t) - z_0} \right) \mathrm{d}t.$$

Hence

$$\left|\operatorname{Ind}_\gamma(w) - \operatorname{Ind}_\gamma(z_0)\right| \leq \frac{1}{2\pi} \int_a^b |\gamma'(t)| \frac{|w - z_0|}{|\gamma(t) - w||\gamma(t) - z_0|} \,\mathrm{d}t \leq \frac{L(\gamma)}{\pi d^2} |w - z_0|.$$

Thus $\operatorname{Ind}_\gamma$ is continuous at z_0. □

Corollary 9.5

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise-C^1 closed curve, then $\operatorname{Ind}_\gamma(z_0)$ is constant on each connected component of $\mathbb{C} \setminus \gamma([a, b])$.

Proof: By Continuity of Winding Number, $\operatorname{Ind}_\gamma$ is continuous and integer-valued. A continuous integer-valued function on a connected set is constant. □

Proposition 9.6 **Winding Number on the Unbounded Component**

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise-C^1 closed curve, then $\operatorname{Ind}_\gamma(z_0) = 0$ for every z_0 in the unbounded connected component of $\mathbb{C} \setminus \gamma([a, b])$.

Proof: Since $\gamma([a, b])$ is compact, there is $R > 0$ such that $|\gamma(t)| \leq R$ for all $t \in [a, b]$. If $|z_0| > 2R$, then

$$|\gamma(t) - z_0| \geq |z_0| - R \geq \frac{|z_0|}{2}.$$

By Integral Formula for Winding Number and the ML Inequality,

$$\left|\operatorname{Ind}_\gamma(z_0)\right| \leq \frac{1}{2\pi} \cdot L(\gamma) \cdot \frac{2}{|z_0|} = \frac{L(\gamma)}{\pi |z_0|}.$$

Taking $|z_0|$ sufficiently large makes the right side less than 1. Since $\operatorname{Ind}_\gamma(z_0) \in \mathbb{Z}$, it follows that $\operatorname{Ind}_\gamma(z_0) = 0$ for all sufficiently large z_0. By the previous corollary, $\operatorname{Ind}_\gamma$ is constant on the unbounded connected component, so it is 0 on that component. □

Theorem 9.7 **Jordan Curve Theorem**

If $\gamma : S^1 \to \mathbb{C}$ is continuous and injective, then $\mathbb{C} \setminus \gamma(S^1)$ has exactly two connected components.

Note

The proof of the Jordan curve theorem is out of scope.

9.3 Contours

Definition 9.4 **Contour**

A *contour* is a formal sum

$$\Gamma = \sum_{j=1}^{m} n_j \gamma_j,$$

where $n_j \in \mathbb{Z}$ and each γ_j is a piecewise-$\mathcal{C}^1$ closed curve. We say that a point lies *on* Γ if it lies on one of the curves γ_j with $n_j \neq 0$.

Definition 9.5 **Integral Over a Contour**

If $\Gamma = \sum_{j=1}^{m} n_j \gamma_j$ is a contour, define

$$\int_\Gamma f(z)\,\mathrm{d}z := \sum_{j=1}^{m} n_j \int_{\gamma_j} f(z)\,\mathrm{d}z.$$

Definition 9.6 **Winding Number of a Contour**

If $\Gamma = \sum_{j=1}^{m} n_j \gamma_j$ is a contour and z_0 does not lie on any of the curves γ_j, define

$$\mathrm{Ind}_\Gamma(z_0) := \sum_{j=1}^{m} n_j \,\mathrm{Ind}_{\gamma_j}(z_0).$$

Definition 9.7 **Simple Contour and Inside**

A contour Γ is *simple* if $\mathrm{Ind}_\Gamma(z)$ is either 0 or 1 for every point z not on Γ. The *inside* of Γ is

$$\mathrm{int}(\Gamma) := \{z \in \mathbb{C} \mid \mathrm{Ind}_\Gamma(z) \neq 0\}.$$

Lemma 9.8 **Separation Lemma**

Suppose $G \subseteq \mathbb{C}$ is open and $K \subseteq G$ is compact. Then there exists a simple contour Γ such that

$$K \subseteq \mathrm{int}(\Gamma) \subseteq G.$$

Proof: Since K is compact and G is open, choose $\varepsilon > 0$ such that

$$\{z \in \mathbb{C} \mid \operatorname{dist}(z, K) < \varepsilon\} \subseteq G.$$

Choose a square grid with side length $\delta > 0$ so small that $\sqrt{2}\delta < \varepsilon$. Let $\mathcal{Q}$ be the finite collection of closed grid squares that meet K, and let

$$U = \cup_{Q \in \mathcal{Q}} Q.$$

Since every point of such a square is within distance at most $\sqrt{2}\delta$ of a point of K, we have $U \subseteq G$.

Also, $K \subseteq U^o$. Indeed, if $z \in K$ lies on a grid line or grid vertex, then all adjacent grid squares that contain z meet K, so their union contains a neighbourhood of z.

Orient each square in $\mathcal{Q}$ counter-clockwise and add their boundary curves as a contour. Whenever two selected squares share an edge, the two copies of that edge have opposite orientations, so cancel them. Let Γ be the resulting contour. Then the range of Γ is contained in ∂U, and hence lies in G.

We claim that Γ is simple and that $\operatorname{int}(\Gamma) = U^o$. For a single positively oriented square Q, the winding number is 1 inside Q and 0 outside Q. Therefore, before cancelling shared edges, the total winding number is the sum of the indicator functions of the interiors of the selected squares. Since distinct grid squares have disjoint interiors, this sum is either 0 or 1.

Cancelling shared edges does not change the contour as a formal sum, so it does not change the winding number at points away from the remaining contour. Thus $\operatorname{Ind}_\Gamma$ is 1 on U^o and 0 on the unbounded side of the boundary. By Continuity of Winding Number, $\operatorname{Ind}_\Gamma$ is constant on each connected component of $\mathbb{C} \setminus \Gamma$, so it is still only 0 or 1 at every point where it is defined. Hence Γ is simple and $\operatorname{int}(\Gamma) = U^o$.

Since $K \subseteq U^o$ and $U \subseteq G$, we get

$$K \subseteq \operatorname{int}(\Gamma) \subseteq G.$$

□

Corollary 9.9 **Cauchy's Formula for the Separating Contour**

Suppose $G \subseteq \mathbb{C}$ is open, $K \subseteq G$ is compact, and Γ is the simple contour constructed in Separation Lemma. If $f : G \to \mathbb{C}$ is holomorphic, then for every $z_0 \in K$,

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

Proof: Fix $z_0 \in K$. Since $K \subseteq \operatorname{int}(\Gamma)$, we have $\operatorname{Ind}_\Gamma(z_0) = 1$. Define

$$h(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0. \end{cases}$$

Then h is holomorphic on G.

Recall that Γ was obtained by adding the positively oriented boundaries of finitely many grid squares and cancelling shared edges. Each selected square lies in G, so Cauchy's theorem for convex regions gives

$$\int_{\partial Q} h(z)\,\mathrm{d}z = 0$$

for every selected square Q. Adding these identities and cancelling the shared edges gives

$$\int_{\Gamma} h(z)\,\mathrm{d}z = 0.$$

Therefore, by Integral Formula for Winding Number,

$$\begin{aligned}\int_{\Gamma} \frac{f(z)}{z-z_0}\,\mathrm{d}z &= \int_{\Gamma} h(z)\,\mathrm{d}z + f(z_0)\int_{\Gamma} \frac{1}{z-z_0}\,\mathrm{d}z \\ &= 2\pi i f(z_0)\,\mathrm{Ind}_{\Gamma}(z_0) = 2\pi i f(z_0).\end{aligned}$$

Dividing by $2\pi i$ proves the result. □

Theorem 9.10 **Cauchy's Theorem**

Suppose $G \subseteq \mathbb{C}$ is open, Γ is a contour in G, and

$$\mathrm{int}(\Gamma) := \{z \in \mathbb{C} \mid \mathrm{Ind}_{\Gamma}(z) \neq 0\} \subseteq G.$$

If $f : G \to \mathbb{C}$ is holomorphic, then

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0.$$

Proof: Let $\mathrm{Range}(\Gamma)$ denote the union of the ranges of the curves occurring in Γ, and set

$$K = \mathrm{Range}(\Gamma) \cup \mathrm{int}(\Gamma).$$

The set K is bounded because the winding number of Γ vanishes on the unbounded component of $\mathbb{C} \setminus \mathrm{Range}(\Gamma)$. It is also closed: away from $\mathrm{Range}(\Gamma)$, the winding number is locally constant, so the complement of K is open. Thus K is compact. By hypothesis, $K \subseteq G$.

By Separation Lemma, there is a simple contour Σ such that

$$K \subseteq \mathrm{int}(\Sigma) \subseteq G.$$

By Cauchy's Formula for the Separating Contour, for every $z \in \mathrm{Range}(\Gamma)$,

$$f(z) = \frac{1}{2\pi i}\int_{\Sigma} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Integrating this identity over Γ and interchanging the finite contour integrals, we obtain

$$\begin{aligned}\int_{\Gamma} f(z)\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{\Sigma} f(w)\left(\int_{\Gamma} \frac{1}{w-z}\,\mathrm{d}z\right)\mathrm{d}w \\ &= -\int_{\Sigma} f(w)\,\mathrm{Ind}_{\Gamma}(w)\,\mathrm{d}w.\end{aligned}$$

Every point w on Σ lies outside K, and hence $\mathrm{Ind}_\Gamma(w) = 0$. Therefore

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

□

10 Residues

Definition 10.1 **Residue**

Suppose z_0 is an isolated singularity of a holomorphic function f. If the Laurent series of f near z_0 is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

then the *residue of f at z_0* is

$$\mathrm{res}(f, z_0) = \mathrm{res}_{z=z_0} f(z) := a_{-1}.$$

Lemma 10.1 **Residue at a Simple Pole**

If f has a simple pole at z_0, then

$$\mathrm{res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof: Since f has a simple pole at z_0, its Laurent series near z_0 has the form

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Therefore

$$(z - z_0) f(z) = a_{-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1},$$

and hence

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1} = \mathrm{res}(f, z_0).$$

□

Exercise 10.1

Compute the residues of the following functions at each of their isolated singularities:

1. $$e^{\frac{1}{z}}$$
2. $$\frac{z^3}{z-1}$$
3. $$\frac{z^3}{(z-1)^3}$$
4. $$\frac{z^k}{(z-a)(z-b)^2}$$
   where $k \geq 0$ and $a \neq b$.
5. $$\frac{\sin z}{2z^2 - 5z + 2}$$
6. $$\frac{2z^2 - 5z + 2}{\sin z}$$

Proposition 10.2 **Integral Formula for the Residue**

Suppose z_0 is an isolated singularity of f, and let C_r be a positively oriented circle centered at z_0 such that f is holomorphic on and inside C_r, except possibly at z_0. Then

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi i} \int_{C_r} f(z)\,\mathrm{d}z.$$

Proof: By Laurent Series Representation, if

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent series of f near z_0, then

$$a_{-1} = \frac{1}{2\pi i} \int_{C_r} f(z)\,\mathrm{d}z.$$

Since $a_{-1} = \operatorname{res}(f, z_0)$, the result follows. □

Theorem 10.3 **Residue Theorem**

Suppose $G \subseteq \mathbb{C}$ is open, Γ is a contour in G with $\operatorname{int}(\Gamma) \subseteq G$, and f is holomorphic on

$$G \setminus \{z_1, \dots, z_p\},$$

where $z_1, \dots, z_p \in \operatorname{int}(\Gamma)$ are isolated singularities of f. Then

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{p} \operatorname{Ind}_\Gamma(z_j) \operatorname{res}(f, z_j).$$

Proof: Choose pairwise disjoint closed disks centered at $z_1, \dots, z_p$ that are contained in G, avoid Γ, and contain no other singularities. Let C_j be the positively oriented boundary of the disk centered at z_j, and define the contour

$$\Lambda = \Gamma - \sum_{j=1}^{p} \operatorname{Ind}_\Gamma(z_j) C_j.$$

We claim that every point in $\operatorname{int}(\Lambda)$ lies in the region where f is holomorphic. Indeed, outside the chosen disks this follows from $\operatorname{int}(\Gamma) \subseteq G$. Inside the disk centered at z_j, both $\operatorname{Ind}_\Gamma$ and Ind_{C_j} are constant, with respective values $\operatorname{Ind}_\Gamma(z_j)$ and 1. Hence

$$\operatorname{Ind}_\Lambda(z) = \operatorname{Ind}_\Gamma(z) - \operatorname{Ind}_\Gamma(z_j) \operatorname{Ind}_{C_j}(z) = 0$$

throughout that disk away from z_j. Thus $\operatorname{int}(\Lambda)$ avoids every singularity of f.

By Cauchy's Theorem,

$$0 = \int_\Lambda f(z)\,\mathrm{d}z = \int_\Gamma f(z)\,\mathrm{d}z - \sum_{j=1}^{p} \operatorname{Ind}_\Gamma(z_j) \int_{C_j} f(z)\,\mathrm{d}z.$$

Applying Integral Formula for the Residue to each C_j gives

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{p} \operatorname{Ind}_\Gamma(z_j) \operatorname{res}(f, z_j).$$

□

Corollary 10.4 **Cauchy's Integral Formula**

Suppose $G \subseteq \mathbb{C}$ is open, Γ is a simple contour in G with $\operatorname{int}(\Gamma) \subseteq G$, and $f : G \to \mathbb{C}$ is holomorphic. Then, for every $z \in \operatorname{int}(\Gamma)$,

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Proof: Fix $z \in \operatorname{int}(\Gamma)$ and define

$$g(\zeta) = \frac{f(\zeta)}{\zeta - z}.$$

Then g has a simple pole at z, and by Residue at a Simple Pole,

$$\operatorname{res}(g, z) = \lim_{\zeta \to z} (\zeta - z) g(\zeta) = f(z).$$

Since Γ is simple, $\operatorname{Ind}_\Gamma(z) = 1$. The result now follows from Residue Theorem. □

Exercise 10.2

Compute

$$\int_C \frac{1}{(z-2)(2z+1)^2(3z-1)^3}\,\mathrm{d}z,$$

where C is the unit circle oriented counter-clockwise.

Hint: Integrate over the positively oriented circle C_R and let $R \to \infty$.

Exercise 10.3

Compute

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$

Exercise 10.4

Compute

$$\int_0^\infty \frac{1}{1+x^a}\,\mathrm{d}x$$

for $a > 1$.

11 Simply Connected Regions

Notation 11.1

The *Riemann sphere* is

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

Definition 11.1 **Simply Connected Region**

A region $G \subseteq \mathbb{C}$ is *simply connected* if every continuous closed curve $\gamma : [0,1] \to G$ can be continuously contracted to a point in G. Explicitly, there are a point $z_0 \in G$ and a continuous map

$$H : [0,1] \times [0,1] \to G$$

such that

$$H(t,0) = \gamma(t), \quad H(t,1) = z_0,$$

and $H(0,s) = H(1,s)$ for all $s, t \in [0,1]$.

Lemma 11.1 **Polygonal Connectedness of Regions**

If $G \subseteq \mathbb{C}$ is a region, then any two points in G can be joined by a piecewise-linear path in G.

Proof: Fix $z_0 \in G$, and let A be the set of points in G that can be joined to z_0 by a piecewise-linear path in G. Clearly A is nonempty. If $z \in G$, choose a disk $D(z, r) \subseteq G$. Every point of $D(z, r)$ can be joined to z by a line segment contained in the disk. It follows that if $z \in A$, then $D(z, r) \subseteq A$, while if $z \notin A$, then $D(z, r) \subseteq G \setminus A$. Thus A and $G \setminus A$ are both open in G. Since G is connected and A is nonempty, $A = G$. $\square$

Theorem 11.2 **Equivalent Forms of Simple Connectivity**

Suppose $G \subseteq \mathbb{C}$ is a region. The following conditions are equivalent:

1. $\widehat{\mathbb{C}} \setminus G$ is connected.
2. If Γ is any contour in G, then $\operatorname{int}(\Gamma) \subseteq G$.
3. If Γ is any contour in G and $f : G \to \mathbb{C}$ is holomorphic, then
$$\int_\Gamma f(z)\, \mathrm{d}z = 0.$$
4. Every holomorphic function $f : G \to \mathbb{C}$ has a primitive on G.
5. Every holomorphic function $f : G \to \mathbb{C}^*$ has a holomorphic logarithm on G; that is, there is a holomorphic function $h : G \to \mathbb{C}$ such that
$$e^{h(z)} = f(z)$$
for every $z \in G$.
6. G is simply connected.

Proof of $1 \Rightarrow 2$***:*** Let Γ be a contour in G. The winding number $\operatorname{Ind}_\Gamma$ is locally constant on $\mathbb{C} \setminus \operatorname{Range}(\Gamma)$ and vanishes on its unbounded component. Extending it to ∞ by setting $\operatorname{Ind}_\Gamma(\infty) = 0$, we obtain a locally constant function on
$$\widehat{\mathbb{C}} \setminus \operatorname{Range}(\Gamma).$$
Since $\widehat{\mathbb{C}} \setminus G$ is connected and is contained in this set, $\operatorname{Ind}_\Gamma$ is constant on $\widehat{\mathbb{C}} \setminus G$. Its value there is 0 because $\infty \in \widehat{\mathbb{C}} \setminus G$. Hence every z with $\operatorname{Ind}_\Gamma(z) \neq 0$ lies in G, so $\operatorname{int}(\Gamma) \subseteq G$. $\square$

Proof of $2 \Rightarrow 3$***:*** This is exactly Cauchy's Theorem. $\square$

Proof of $3 \Rightarrow 4$***:*** Fix $z_0 \in G$. By Polygonal Connectedness of Regions, for every $z \in G$ there is a piecewise-linear path γ_z in G from z_0 to z. Define
$$F(z) = \int_{\gamma_z} f(w)\, \mathrm{d}w.$$

This is independent of the chosen path: if γ_1 and γ_2 both join z_0 to z, then $\gamma_1 - \gamma_2$ is a closed curve in G, so condition 3 gives

$$\int_{\gamma_1} f(w)\,\mathrm{d}w - \int_{\gamma_2} f(w)\,\mathrm{d}w = 0.$$

Fix $z \in G$ and choose a disk $D(z,r) \subseteq G$. If $w \in D(z,r)$, path independence allows us to compute $F(w)$ using a path to z followed by the line segment $[z,w]$. Hence

$$\frac{F(w)-F(z)}{w-z} = \frac{1}{w-z}\int_{[z,w]} f(\zeta)\,\mathrm{d}\zeta.$$

Parametrizing the segment by $z+t(w-z)$ gives

$$\frac{F(w)-F(z)}{w-z} = \int_0^1 f(z+t(w-z))\,\mathrm{d}t.$$

As $w \to z$, the right-hand side tends to $f(z)$ by continuity. Therefore $F'(z) = f(z)$, so F is a primitive of f on G. □

Proof of $4 \Rightarrow 5$***:*** Let $f : G \to \mathbb{C}^*$ be holomorphic. Since f never vanishes, $\frac{f'}{f}$ is holomorphic on G. By condition 4, there is a holomorphic function $g : G \to \mathbb{C}$ such that

$$g' = \frac{f'}{f}.$$

It follows that

$$(fe^{-g})' = e^{-g}(f' - fg') = 0,$$

so $fe^{-g} = c$ for some $c \in \mathbb{C}^*$. Choose $a \in \mathbb{C}$ such that $e^a = c$. Then $h = g + a$ is holomorphic and

$$e^h = e^a e^g = ce^g = f.$$

Thus h is a holomorphic logarithm of f. □

Proof of $5 \Rightarrow 1$***:*** We prove the contrapositive. Suppose $\widehat{\mathbb{C}} \setminus G$ is disconnected. Write

$$\widehat{\mathbb{C}} \setminus G = A \cup B,$$

where A and B are disjoint nonempty compact sets with $\infty \in A$. Then $B \subseteq \mathbb{C}$ is bounded. As in the square-grid construction of Separation Lemma, choose all squares in a sufficiently fine grid that meet B, and add their positively oriented boundaries. Since A and B are disjoint compact sets, the grid can be chosen so that the resulting simple contour Γ lies in G. Moreover,

$$\mathrm{Ind}_\Gamma(a) = 1$$

for every $a \in B$.

Fix $a \in B$. The function $f(z) = z - a$ is holomorphic and nonzero on G. If condition 5 held, there would be a holomorphic function $h : G \to \mathbb{C}$ such that

$$e^{h(z)} = z - a.$$

Differentiating gives $h'(z) = \frac{1}{z-a}$. The fundamental theorem for path integrals would then imply

$$0 = \int_\Gamma h'(z)\,\mathrm{d}z = \int_\Gamma \frac{1}{z-a}\,\mathrm{d}z = 2\pi i \operatorname{Ind}_\Gamma(a) = 2\pi i,$$

a contradiction. Therefore condition 5 fails. □

Proof of $6 \Rightarrow 5$***:*** The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map. Since G is simply connected, every continuous map $f : G \to \mathbb{C}^*$ lifts through exp after choosing the value of the lift at one point. Thus there is a continuous function $h : G \to \mathbb{C}$ such that

$$e^h = f.$$

Locally, h differs by a constant in $2\pi i\mathbb{Z}$ from a holomorphic branch of the logarithm of f, so h is holomorphic. Hence condition 5 holds. □

Proof of $1 \Rightarrow 6$***:*** If $G = \mathbb{C}$, then G is simply connected. Otherwise, the Riemann Mapping Theorem gives a biholomorphism from G to $\mathbb{D}$. Since $\mathbb{D}$ is simply connected and simple connectivity is preserved by homeomorphisms, G is simply connected. □

Theorem 11.3 **Riemann Mapping Theorem**

Suppose $G \subsetneq \mathbb{C}$ is a region and $\widehat{\mathbb{C}} \setminus G$ is connected. Then either $G = \mathbb{C}$, or there is a biholomorphism

$$\varphi : G \to \mathbb{D}.$$

Note

The proof of the Riemann mapping theorem is out of scope.

12 Linear Fractional Transformations

12.1 Projective Lines

Definition 12.1 **Real Projective Line**

The *real projective line* $\mathbb{RP}^1$ is the set of one-dimensional linear subspaces of $\mathbb{R}^2$.

Remark 12.1

Every line in $\mathbb{R}^2$ other than $\operatorname{span}\{(1,0)\}$ contains a unique point of the form $(x,1)$. Thus

$$\mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\},$$

where $x \in \mathbb{R}$ corresponds to $\operatorname{span}\{(x,1)\}$ and ∞ corresponds to $\operatorname{span}\{(1,0)\}$. Geometrically, $\mathbb{RP}^1$ is also homeomorphic to $\mathbb{S}^1$.

Definition 12.2 **Complex Projective Line**

The *complex projective line* $\mathbb{CP}^1$ is the set of one-dimensional complex-linear subspaces of $\mathbb{C}^2$. Equivalently,

$$\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{0\})/\sim\,,$$

where

$$(z_1, w_1) \sim (z_2, w_2)$$

if $(z_2, w_2) = \lambda(z_1, w_1)$ for some $\lambda \in \mathbb{C}^*$. The equivalence class of (z, w) is denoted by $[z : w]$ and is called a *homogeneous coordinate*.

Proposition 12.1 **The Projective Line as the Riemann Sphere**

The map

$$\begin{aligned} \widehat{\mathbb{C}} &\longrightarrow \mathbb{CP}^1 \\ z &\longmapsto [z : 1] \\ \infty &\longmapsto [1 : 0] \end{aligned}$$

is a bijection. Thus

$$\mathbb{CP}^1 \cong \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Proof: If $[z : w] \in \mathbb{CP}^1$ and $w \neq 0$, then

$$[z : w] = \left[\frac{z}{w} : 1\right].$$

If $w = 0$, then $z \neq 0$ and $[z : 0] = [1 : 0]$. These two cases are mutually exclusive, and the displayed representatives are unique. □

12.2 Induced Projective Transformations

Proposition 12.2 **Projectivization of an Invertible Linear Map**

Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be invertible, with matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc \neq 0.$$

Then T induces a bijection

$$\begin{aligned} \varphi_T : \mathbb{CP}^1 &\longrightarrow \mathbb{CP}^1 \\ [z : w] &\longmapsto [az + bw : cz + dw]. \end{aligned}$$

Proof: If $[z : w] = [\lambda z : \lambda w]$ for some $\lambda \in \mathbb{C}^*$, then

$$T\begin{pmatrix}\lambda z\\ \lambda w\end{pmatrix} = \lambda T\begin{pmatrix}z\\ w\end{pmatrix},$$

so the image does not depend on the chosen representative. Since T is invertible, $T(z,w) \neq (0,0)$ whenever $(z,w) \neq (0,0)$, and $\varphi_{T^{-1}}$ is the inverse of φ_T. $\square$

Definition 12.3 **Linear Fractional Transformation**

Let $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$. The *linear fractional transformation*, or *Möbius transformation*, associated with the matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is the map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$\varphi(z) = \begin{cases} \frac{az+b}{cz+d} & z \in \mathbb{C} \text{ and } cz+d \neq 0 \\ \infty & z \in \mathbb{C} \text{ and } cz+d = 0 \\ \frac{a}{c} & z = \infty \text{ and } c \neq 0 \\ \infty & z = \infty \text{ and } c = 0. \end{cases}$$

Equivalently, under the identification in The Projective Line as the Riemann Sphere,

$$\varphi([z : w]) = [az + bw : cz + dw].$$

Note

The determinant condition ensures that the cases above are well-defined. In particular, if $cz + d = 0$, then $az + b \neq 0$, so the corresponding homogeneous coordinate is $[1 : 0] = \infty$. Also, if $c = 0$, then $a, d \neq 0$, so φ fixes ∞.

Example 12.1

The following transformations illustrate the cases involving ∞:

1. If $\varphi(z) = z + b$, then $\varphi(\infty) = \infty$.
2. If $\varphi(z) = \lambda z$ with $\lambda \neq 0$, then $\varphi(0) = 0$ and $\varphi(\infty) = \infty$.
3. If $\varphi(z) = \frac{1}{z}$, then
$$\varphi(0) = \infty \quad \text{and} \quad \varphi(\infty) = 0.$$
4. If $c \neq 0$, then
$$\varphi\left(-\frac{d}{c}\right) = \infty \quad \text{and} \quad \varphi(\infty) = \frac{a}{c}.$$

12.3 The Möbius Group

Proposition 12.3 **The Möbius Group**

The set $\mathcal{M}$ of linear fractional transformations is a group under composition. The map

$$\begin{aligned} \Phi : \mathrm{GL}_2(\mathbb{C}) &\longrightarrow \mathcal{M} \\ T &\longmapsto \varphi_T \end{aligned}$$

is a surjective group homomorphism with kernel

$$\ker \Phi = \mathbb{C}^* I_2.$$

Consequently,

$$\mathcal{M} \cong \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^* I_2 =: \mathrm{PGL}_2(\mathbb{C}).$$

Proof: For $S, T \in \mathrm{GL}_2(\mathbb{C})$, projectivization gives

$$\varphi_{ST} = \varphi_S \circ \varphi_T,$$

so Φ is a homomorphism. It is surjective by the definition of a linear fractional transformation.

Every nonzero scalar matrix fixes each projective line, so it lies in $\ker \Phi$. Conversely, suppose φ_T is the identity. Then every nonzero vector is an eigenvector of T. In particular, if e_1, e_2 is the standard basis, write

$$Te_1 = \lambda_1 e_1 \quad \text{and} \quad Te_2 = \lambda_2 e_2.$$

Since $e_1 + e_2$ is also an eigenvector, we must have $\lambda_1 = \lambda_2$. Hence $T = \lambda I_2$ for some $\lambda \in \mathbb{C}^*$. □

Remark 12.2

Over $\mathbb{C}$, this group is also isomorphic to

$$\mathrm{PSL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C})/\{\pm I_2\}.$$

Indeed, every invertible matrix can be multiplied by a scalar so that its determinant is 1, and the two choices of square root differ by a sign.

Proposition 12.4 **Fixed Points of a Möbius Transformation**

Every linear fractional transformation other than the identity has exactly one or two fixed points in $\widehat{\mathbb{C}}$.

Proof: Let $\varphi = \varphi_T$. A point of $\mathbb{CP}^1$ is fixed by φ_T exactly when its corresponding one-dimensional subspace is invariant under T, or equivalently, when it is an eigenline of T.

Every 2×2 complex matrix has an eigenvalue. If T has two distinct eigenlines, then φ_T has two fixed points. Otherwise, it has exactly one eigenline and hence exactly one fixed point. The only remaining possibility is that every line is an eigenline, but then T is a scalar matrix by the argument in The Möbius Group, so φ_T is the identity. □

13 Harmonic Functions

13.1 Definition and Basic Properties

Definition 13.1 **Harmonic Function**

Let $G \subseteq \mathbb{C} \cong \mathbb{R}^2$ be open. A function $f : G \to \mathbb{C}$ is *harmonic* if it is of class $\mathcal{C}^2$ and satisfies *Laplace's equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Note

A function $f : G \to \mathbb{C}$ is harmonic if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic.

Example 13.1

The functions

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy$$

are harmonic on $\mathbb{C}$. In fact,

$$u(x, y) + iv(x, y) = (x + iy)^2.$$

Proposition 13.1 **Holomorphic Functions Are Harmonic**

If $G \subseteq \mathbb{C}$ is open and $f : G \to \mathbb{C}$ is holomorphic, then f is harmonic.

Proof: Write $f = u + iv$. By Analyticity of Holomorphic Functions, the functions u and v are of class $\mathcal{C}^2$. The Cauchy–Riemann equations give

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Differentiating and using equality of mixed partial derivatives, we obtain

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

and

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$$

Thus both u and v are harmonic, and hence so is f. □

13.2 Harmonic Conjugates

Definition 13.2 **Harmonic Conjugate**

Let $u : G \to \mathbb{R}$ be harmonic. A function $v : G \to \mathbb{R}$ is a *harmonic conjugate* of u if

$$f(x + iy) = u(x, y) + iv(x, y)$$

defines a holomorphic function on G.

Note

If v is a harmonic conjugate of u, then v is harmonic by Holomorphic Functions Are Harmonic. On a region, a harmonic conjugate is unique up to an additive constant.

Theorem 13.2 **Existence of Harmonic Conjugates**

If $G \subseteq \mathbb{C}$ is simply connected, then every harmonic function $u : G \to \mathbb{R}$ has a harmonic conjugate.

Proof: Define

$$f = u_x - iu_y.$$

We first show that f is holomorphic. The real and imaginary parts of f are u_x and $-u_y$. Since u is harmonic,

$$(u_x)_x = u_{xx} = -u_{yy} = (-u_y)_y,$$

while equality of mixed partial derivatives gives

$$(u_x)_y = u_{xy} = -(-u_y)_x.$$

Thus f satisfies the Cauchy–Riemann equations and is holomorphic.

Since G is simply connected, Equivalent Forms of Simple Connectivity implies that f has a holomorphic primitive $F = U + iV$. Hence

$$F' = u_x - iu_y.$$

On the other hand, the Cauchy–Riemann equations for F give

$$F' = U_x - iU_y.$$

Therefore $U_x = u_x$ and $U_y = u_y$, so $U - u$ is constant on G. Subtracting this real constant from F, we may assume that $U = u$. It follows that

$$F = u + iV,$$

so V is a harmonic conjugate of u. $\square$

Example 13.2

Let

$$G = \{z \in \mathbb{C} \mid 0 < |z| < R\}$$

and define

$$u(x,y) = \frac{1}{2}\log(x^2 + y^2) = \log|z|.$$

Then u is harmonic on G, but it has no harmonic conjugate on all of G. Indeed,

$$u_x - iu_y = \frac{1}{z}.$$

If u had a harmonic conjugate, then $\frac{1}{z}$ would have a primitive on G. But for the positively oriented circle C_r centered at 0 with $0 < r < R$,

$$\int_{C_r} \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0,$$

a contradiction. Locally, the harmonic conjugates of $u = \log|z|$ are branches of $\arg z$.

Corollary 13.3 **Harmonic Conjugates and Simple Connectivity**

Let $G \subseteq \mathbb{C}$ be a region. Then G is simply connected if and only if every harmonic function $u : G \to \mathbb{R}$ has a harmonic conjugate.

Proof: If G is simply connected, the result follows from Existence of Harmonic Conjugates.

Conversely, suppose every harmonic function on G has a harmonic conjugate. Let Γ be a contour in G, and fix $a \in \mathbb{C} \setminus G$. The function

$$u_a(z) = \log|z - a|$$

is harmonic on G: near each point of G, it is the real part of a holomorphic branch of $\log(z - a)$. By hypothesis, u_a has a harmonic conjugate, so $u_a + iv$ is holomorphic for some $v : G \to \mathbb{R}$. Its derivative is

$$(u_a + iv)' = (u_a)_x - i(u_a)_y = \frac{1}{z - a}.$$

Therefore, by the fundamental theorem for path integrals,

$$2\pi i \operatorname{Ind}_\Gamma(a) = \int_\Gamma \frac{1}{z - a}\,\mathrm{d}z = 0.$$

Hence $\operatorname{Ind}_\Gamma(a) = 0$ for every $a \notin G$, so $\operatorname{int}(\Gamma) \subseteq G$. Since this holds for every contour Γ in G, Equivalent Forms of Simple Connectivity implies that G is simply connected. □

Corollary 13.4 **Identity Theorem for Harmonic Functions**

Let $G \subseteq \mathbb{C}$ be a region, and suppose $u, v : G \to \mathbb{R}$ are harmonic. If $u = v$ on a nonempty open subset of G, then $u = v$ on G.

Proof: Set $w = u - v$, and let

$$A = \{z \in G \mid w \text{ vanishes on a neighbourhood of } z\}.$$

By hypothesis, A is nonempty, and it is open by definition. We show that it is closed in G.

Let z_0 lie in the closure of A relative to G, and choose a disk $D = D(z_0, r)$ with $D \subseteq G$. Since D is simply connected, Existence of Harmonic Conjugates gives a harmonic conjugate $\tilde{w}$ of w on D. Thus

$$F = w + i\tilde{w}$$

is holomorphic on D. The disk D meets A, so there is a nonempty open subset $U \subseteq D$ on which $w = 0$. The real part of F is therefore constant on U, and the Cauchy–Riemann equations imply that $F' = 0$ on U. By Identity Theorem, $F' = 0$ throughout D. Hence F is constant on D, and its real part w is 0 on D. Thus $z_0 \in A$.

Therefore A is both open and closed in the connected set G. Since A is nonempty, $A = G$, so $u = v$ on G. □