Pointwise Limits of Differential-Operator Iterates

Analytic Input and a Singular Counterexample

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Abstract. Let L be a linear differential operator of positive order with real-analytic coefficients on an open interval, allowing zeros of the leading coefficient. For real-analytic input f, we prove that pointwise boundedness of $(L^n f)$ implies local uniform bounds on every derivative of the iterates. If the orbit converges pointwise, its limit g is real analytic, the convergence holds locally uniformly with all derivatives, and $Lg = g$. For one such first-order operator with polynomial coefficients, we construct a smooth, nonanalytic input whose iterates converge to the indicator of the origin.

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1 Introduction

1.1 Background and results

Our previous work (Edmonstone and Tchouikine 2026) studied the following question of Paul Cusson (Tao 2022).

Problem **Pointwise limits of successive derivatives**

Suppose $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(x) \to g(x)$ for every real x. Must g be differentiable, and must $g' = g$?

Tao's analyticity theorem (Tao 2022) gives an affirmative answer: pointwise boundedness of all derivatives forces an entire extension of f and local uniform derivative bounds. These bounds justify passing to the limit in $f^{(n+1)} = Df^{(n)}$.

The first-order theorem (Edmonstone and Tchouikine 2026, Section 2) treats $Lh = ah' + bh$, where $a, b \in C^\infty(\mathbb{R})$ and a is nowhere zero. A change of variable and an integrating factor turn repeated application of L into repeated ordinary differentiation. For $f \in C^\infty(\mathbb{R})$, pointwise convergence of $L^n f$ therefore implies that the limit is smooth and satisfies $Lg = g$.

In Theorem 3.3 we consider operators of arbitrary positive order with analytic coefficients and input, allowing zeros of the leading coefficient. Pointwise boundedness of the orbit gives local uniform bounds on every derivative by Proposition 3.2; pointwise convergence then gives a real-analytic limit satisfying $Lg = g$. The proof obtains bounds on one real interval, propagates them along complex paths avoiding the zeros, and uses the maximum-modulus principle to extend the bounds across those zeros.

In Theorem 4.3, a first-order operator with polynomial coefficients and a single zero of its leading coefficient has a smooth input whose orbit converges to a discontinuous limit. This shows that the analytic-input assumption in Theorem 3.3 cannot be replaced by smoothness. It also shows that the nowhere-zero hypothesis in the smooth first-order theorem cannot be omitted.

1.2 Notation

Throughout, $I \subseteq \mathbb{R}$ is a nonempty open interval. Inputs and coefficients on I are real valued. Holomorphic extensions are denoted by the same symbols as the original functions. Write $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $D = \frac{\mathrm{d}}{\mathrm{d}x}$, and $h^{(0)} = h$. For an operator

$$Lh = \sum_{j=0}^{m} a_j h^{(j)},$$

write $L^0 f = f$. Thus $L(L^n f) = L^{n+1} f$ for $n \in \mathbb{N}_0$.

For a continuous function h on a nonempty compact set K, put $\|h\|_K = \sup_{x \in K} |h(x)|$. We say that v_n converges to g **locally uniformly with all derivatives** if g is smooth and

$$\left\| v_n^{(q)} - g^{(q)} \right\|_K \xrightarrow[n\to\infty]{} 0 \quad \text{for every } q \in \mathbb{N}_0 \text{ and every compact interval } K \subseteq I.$$

We write $v_n \to g$ in $C^\infty_{\mathrm{loc}}(I)$ for this convergence. In the complex plane, $D(z, R) = \{w \in \mathbb{C} \mid |w - z| < R\}$ denotes an open disk, and $\overline{D(z, R)}$ its closure.

2 Elementary estimates

2.1 Interpolation

Lemma 2.1 **Scaled interpolation**

Fix $m \in \mathbb{N}$. There is a constant C_m, depending only on m, such that for every compact interval $J = [\alpha, \beta]$ with $\alpha < \beta$, every real function h of class C^m on a neighborhood of J, and $0 < \delta \le \frac{\beta - \alpha}{2m}$,

$$\left\| h^{(j)} \right\|_J \le C_m \delta^{-j} \|h\|_J + C_m \delta^{m-j} \left\| h^{(m)} \right\|_J \quad (0 \le j < m).$$

Proof: Fix $x \in J$. Choose $\varepsilon = 1$ if $x \le \frac{\alpha+\beta}{2}$, and $\varepsilon = -1$ otherwise. Then all points $x + \varepsilon \ell \delta$, $1 \le \ell \le m$, lie in J. Taylor's theorem gives

$$h(x + \varepsilon\ell\delta) = \sum_{j=0}^{m-1} \frac{(\varepsilon\ell)^j}{j!} \delta^j h^{(j)}(x) + R_\ell,$$

where

$$|R_\ell| \le \frac{(m\delta)^m}{m!} \left\| h^{(m)} \right\|_J.$$

These are m linear equations for the unknowns $\delta^j h^{(j)}(x)$, $0 \le j < m$. Their coefficient matrix has entries $V_{\ell,j} = \frac{(\varepsilon\ell)^j}{j!}$.

If $Vv = 0$, the polynomial $p(t) = \sum_{j=0}^{m-1} \frac{v_j}{j!} t^j$ vanishes at the m distinct points $\varepsilon, 2\varepsilon, \ldots, m\varepsilon$. Since its degree is at most $m - 1$, it is zero. Thus V is invertible.

Let A_m be the maximum sum of absolute values of entries in a row of V^{-1}, over both choices of ε. Multiplication by V^{-1} gives

$$\left|h^{(j)}(x)\right| \le A_m \delta^{-j}\left(\|h\|_J + \frac{(m\delta)^m}{m!}\left\|h^{(m)}\right\|_J\right).$$

Take the supremum over $x \in J$ and set $C_m = A_m \max\{1, \frac{m^m}{m!}\}$. □

We will also use the following first-derivative estimate. Let J be a compact interval and let h be twice continuously differentiable on a neighborhood of J. If $t > 0$ and $[x, x+t] \subseteq J$, Taylor's theorem gives

$$|h'(x)| \le \frac{2}{t}\|h\|_J + \frac{t}{2}\|h''\|_J.$$

Indeed, subtract $h(x)$ from the formula for $h(x+t)$ and divide by t.

2.2 Smooth convergence

Lemma 2.2 **Pointwise convergence and local derivative bounds**

Suppose $v_n \in C^\infty(I)$ and $v_n(x) \to g(x)$ for every $x \in I$. Assume that for every compact interval $J \subseteq I$ and every $j \in \mathbb{N}_0$ there is a finite constant $B_{J,j}$ such that

$$\left\|v_n^{(j)}\right\|_J \le B_{J,j} \quad \text{for every } n.$$

Then g is smooth and $v_n \to g$ locally uniformly with all derivatives.

Proof: **First, the functions converge uniformly on compact intervals.** Fix a compact interval $K \subseteq I$ and let $B = B_{K,1}$. The mean value theorem gives

$$|v_n(x) - v_n(y)| \le B|x - y| \quad (x, y \in K).$$

Given $\varepsilon > 0$, choose $\delta > 0$ with $2B\delta < \frac{\varepsilon}{2}$. Cover K by finitely many intervals of radius δ centered at points $x_1, \ldots, x_q$ of K. At these finitely many points, pointwise convergence gives one N such that

$$|v_n(x_i) - v_m(x_i)| < \frac{\varepsilon}{2} \quad (n, m \ge N, 1 \le i \le q).$$

For any $x \in K$, choose x_i within δ of x. The triangle inequality gives

$$|v_n(x) - v_m(x)| < 2B\delta + \frac{\varepsilon}{2} < \varepsilon.$$

Thus (v_n) is uniformly Cauchy on K. Letting m tend to infinity shows uniform convergence to its pointwise limit g. In particular, g is continuous.

Next, the derivatives are uniformly Cauchy as well. Suppose that $\left(v_n^{(j)}\right)$ is uniformly Cauchy on every compact interval. Choose $K = [c, d]$ and a larger $J = [a, b] \subseteq I$ with $a < c \leq d < b$. Apply the preceding first-derivative estimate to $h = v_n^{(j)} - v_m^{(j)}$. For $0 < t < b - d$ it gives

$$\left\| v_n^{(j+1)} - v_m^{(j+1)} \right\|_K \leq \frac{2}{t} \left\| v_n^{(j)} - v_m^{(j)} \right\|_J + tB_{J,j+2}.$$

Given $\varepsilon > 0$, first fix t small enough that the last term is less than $\frac{\varepsilon}{2}$. The induction hypothesis then makes the first term less than $\frac{\varepsilon}{2}$ for all sufficiently large n, m. This proves the induction step. Starting at $j = 0$, all derivative sequences converge uniformly on compact intervals. Denote their continuous limits by g_j, with $g_0 = g$.

Finally, identify these limits as derivatives. For $x, y \in I$, the fundamental theorem of calculus gives

$$v_n^{(j)}(y) - v_n^{(j)}(x) = \int_x^y v_n^{(j+1)}(s)\,\mathrm{d}s.$$

Uniform convergence on the interval between x and y permits passage to the limit, so

$$g_j(y) - g_j(x) = \int_x^y g_{j+1}(s)\,\mathrm{d}s.$$

Since g_{j+1} is continuous, this identity implies ${g_j}' = g_{j+1}$. Induction gives $g_j = g^{(j)}$ for every j. □

3 Analytic coefficients and input

3.1 Uniform propagation on a complex disk

Away from a zero of the leading coefficient, the relation $L(L^n f) = L^{n+1} f$ expresses $(L^n f)^{(m)}$ in terms of $L^{n+1} f$ and the lower derivatives of $L^n f$. The following lemma propagates a bound for these functions from the center of a disk to a smaller disk. The smaller radius depends only on the coefficients and the original radius, so it can be used uniformly along a fixed path.

Lemma 3.1 **Uniform propagation**

Fix $m \in \mathbb{N}$, a radius $R > 0$, and $z_* \in \mathbb{C}$. Suppose that $c, b_0, \dots, b_{m-1}$ and the functions $v_{n,j}$, $n \in \mathbb{N}_0$, $0 \le j < m$, are holomorphic on a common neighborhood of $\overline{D(z_*, R)}$ and satisfy

$$v'_{n,j} = v_{n,j+1} \quad (0 \le j < m-1),$$
$$v'_{n,m-1} = c v_{n+1,0} + \sum_{j=0}^{m-1} b_j v_{n,j}.$$

Put $E = \overline{D(z_*, R)}$. Choose

$$C_0 \ge \max\left\{1, \|c\|_E + \sum_{j=0}^{m-1} \|b_j\|_E\right\}, \quad A = \max\left\{\frac{2}{R}, 2C_0\right\}.$$

If a finite constant $M \ge 0$ satisfies $|v_{n,j}(z_*)| \le M$ for all n, j, then

$$|v_{n,j}(z)| \le 2M \quad \text{whenever } |z - z_*| \le \frac{1}{2A},$$

for every $n \in \mathbb{N}_0$ and $0 \le j < m$.

Proof: Expand at z_*:

$$v_{n,j}(z) = \sum_{k=0}^{\infty} d_{n,j,k}(z - z_*)^k,$$
$$c(z) = \sum_{\ell=0}^{\infty} c_\ell (z - z_*)^\ell, \quad b_j(z) = \sum_{\ell=0}^{\infty} b_{j,\ell}(z - z_*)^\ell.$$

Cauchy's estimates give

$$|c_\ell| + \sum_{j=0}^{m-1} |b_{j,\ell}| \le C_0 R^{-\ell}.$$

Set $T_k = \sup_{n \in \mathbb{N}_0, 0 \le j < m} |d_{n,j,k}|$, initially allowing the value ∞. The hypothesis gives $T_0 \le M$. Comparing coefficients in the differential relations, we obtain

$$(k+1) d_{n,j,k+1} = d_{n,j+1,k} \quad (j < m-1),$$
$$(k+1) d_{n,m-1,k+1} = \sum_{\ell=0}^{k} \left(c_\ell d_{n+1,0,k-\ell} + \sum_{j=0}^{m-1} b_{j,\ell} d_{n,j,k-\ell} \right).$$

Consequently, whenever $T_0, \dots, T_k$ are finite,

$$(k+1)T_{k+1} \le C_0 \sum_{\ell=0}^{k} R^{-\ell} T_{k-\ell}.$$

For the first set of relations, the $\ell = 0$ term suffices, since $C_0 \ge 1$. For the last relation, use the coefficient bound above. This also proves inductively that every T_k is finite.

We claim that $T_k \le MA^k$ for all $k \in \mathbb{N}_0$. It holds at $k = 0$. Assuming it through degree k and using $RA \ge 2$, we get

$$\begin{aligned}(k+1)T_{k+1} &\le C_0 M A^k \sum_{\ell=0}^{k} (RA)^{-\ell} \\ &\le 2C_0 M A^k \le M A^{k+1}.\end{aligned}$$

Since $k + 1 \ge 1$, this proves the induction step. The Taylor series of every $v_{n,j}$ represents it on $D(z_*, R)$, and $\frac{1}{2A} \le \frac{R}{4}$. Hence, for $|z - z_*| \le \frac{1}{2A}$,

$$\left|v_{n,j}(z)\right| \le \sum_{k=0}^{\infty} M A^k |z - z_*|^k \le 2M.$$

□

3.2 Local derivative estimates

Proposition 3.2 **Local derivative estimates**

Let $m \in \mathbb{N}$ and define

$$L = \sum_{j=0}^{m} a_j D^j,$$

where $a_0, \dots, a_m$ are real analytic on I and a_m is not identically zero. Let f be real analytic on I and suppose that

$$\sup_{n \in \mathbb{N}_0} |(L^n f)(x)| < \infty \quad \text{for every } x \in I.$$

Then, for every compact interval $K \subseteq I$, there are constants $C, r > 0$ such that

$$\left\| (L^n f)^{(q)} \right\|_K \le C q! r^{-q} \quad (n, q \in \mathbb{N}_0).$$

Proof:

Claim 1: There exist $x_* \in I$ with $a_m(x_*) \neq 0$ and $M_* > 0$ such that

$$\left|(L^n f)^{(j)}(x_*)\right| \leq M_* \quad \text{for all } n \in \mathbb{N}_0, 0 \leq j < m.$$

Proof of Claim 1: Since a_m is not identically zero, there is a compact interval $J_0 = [\alpha_0, \beta_0] \subseteq I$, with $\alpha_0 < \beta_0$, on which a_m does not vanish. For each $N \in \mathbb{N}$, define

$$E_N := \{x \in J_0 \mid |(L^n f)(x)| \leq N \text{ for every } n \in \mathbb{N}_0\}.$$

Note that E_N is closed and $\{E_n\}_{n\in\mathbb{N}}$ covers J_0. Baire Category Theorem gives an $N \in \mathbb{N}$ with $\operatorname{int}_{J_0}(E_N) \neq \varnothing$. Hence there are $x \in J_0$ and $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \cap J_0 \subseteq E_N.$$

Since J_0 is a nondegenerate interval, this intersection is an interval of positive length. It therefore contains a compact interval $J = [\alpha, \beta] \subseteq E_N$ with $\alpha < \beta$. Thus $\|L^n f\|_J \leq N$ for all $n \in \mathbb{N}_0$.

On J, the equation $L(L^n f) = L^{n+1} f$ gives

$$(L^n f)^{(m)} = \frac{L^{n+1} f}{a_m} - \sum_{j=0}^{m-1} \frac{a_j}{a_m} (L^n f)^{(j)}.$$

Since f and the coefficients of L are real analytic on I, every $L^n f$ is real analytic on I, hence belongs to $C^m(I)$. Since $J \subseteq I$ and I is open, Lemma 2.1 applies to each $L^n f$. For $0 < \delta \leq \frac{\beta-\alpha}{2m}$, this yields

$$\begin{aligned}\left\|(L^n f)^{(m)}\right\|_J \leq N\left\|\frac{1}{a_m}\right\|_J + C_m N \sum_{j=0}^{m-1} \left\|\frac{a_j}{a_m}\right\|_J \delta^{-j} \\ + C_m \left\|(L^n f)^{(m)}\right\|_J \sum_{j=0}^{m-1} \left\|\frac{a_j}{a_m}\right\|_J \delta^{m-j}.\end{aligned}$$

All powers $m - j$ in the last sum are positive. Choose a fixed $\delta > 0$ small enough that the coefficient of $\left\|(L^n f)^{(m)}\right\|_J$ on the right is at most $\frac{1}{2}$. This norm is finite for each n, since $(L^n f)^{(m)}$ is continuous on the compact interval J. Subtracting $\frac{1}{2}\left\|(L^n f)^{(m)}\right\|_J$ from both sides of the resulting inequality gives

$$\frac{1}{2}\left\|(L^n f)^{(m)}\right\|_J \leq N\left\|\frac{1}{a_m}\right\|_J + C_m N \sum_{j=0}^{m-1} \left\|\frac{a_j}{a_m}\right\|_J \delta^{-j}.$$

Multiplying both sides by 2, we obtain

$$\left\|(L^n f)^{(m)}\right\|_J \leq 2N\left\|\frac{1}{a_m}\right\|_J + 2C_m N \sum_{j=0}^{m-1} \left\|\frac{a_j}{a_m}\right\|_J \delta^{-j}.$$

This bound is independent of n. Applying Lemma 2.1 again bounds $\left\|(L^n f)^{(j)}\right\|_J$ uniformly in n for every $0 \leq j < m$. In particular, at a chosen $x_* \in (\alpha, \beta)$,

$$\left|(L^n f)^{(j)}(x_*)\right| \leq M_* \quad \text{for all } n \in \mathbb{N}_0, 0 \leq j < m$$

for some $M_* > 0$. Since a_m does not vanish on J, we also have $a_m(x_*) \neq 0$. □

Claim 2. There is a connected open set $\Omega \subseteq \mathbb{C}$ containing I on which $f, a_0, ..., a_m$ and every iterate $L^n f$ have holomorphic extensions.

Proof of Claim 2: Fix $x \in I$. For each $h \in \{f, a_0, ..., a_m\}$, real analyticity gives $R_h > 0$ such that $(x - R_h, x + R_h) \subseteq I$ and

$$h(t) = \sum_{k=0}^{\infty} c_{h,k}(t - x)^k \quad \text{for } |t - x| < R_h.$$

Choose $0 < \rho_h < R_h$. Since the series converges at $t = x + \rho_h$, its terms are bounded, so there is $M_h > 0$ such that $\left|c_{h,k}\right|\rho_h^k \leq M_h$ for all $k \in \mathbb{N}_0$. For complex z with $|z - x| < \rho_h$, we therefore have

$$\left|c_{h,k}(z - x)^k\right| \leq M_h\left(\frac{|z - x|}{\rho_h}\right)^k.$$

Comparison with a geometric series shows that $\sum_{k=0}^{\infty} c_{h,k}(z - x)^k$ converges absolutely on $D(x, \rho_h)$. By the theorem on differentiation of power series, its sum is holomorphic on this disk and agrees with h on its real diameter. Since there are only finitely many functions $f, a_0, ..., a_m$, we can choose $r_x > 0$ smaller than all their radii ρ_h. Thus they have holomorphic extensions on the common disk $D(x, r_x)$, whose real diameter is contained in I.

Let $\Omega = \bigcup_{x \in I} D(x, r_x)$. This is an open set containing I. If two of these disks intersect, their intersection contains a real interval. The extensions agree on that interval, hence on the intersection by the identity theorem. They therefore define holomorphic functions on Ω. The interval I is connected, and every disk $D(x, r_x)$ is connected and contains its center $x \in I$. Thus

$$\Omega = I \cup \bigcup_{x \in I} D(x, r_x)$$

is connected as it is the union of the connected set I with connected sets that each have nonempty intersection with I.

Differentiation and multiplication preserve holomorphy on Ω, so all iterates $L^n f$ are holomorphic on this same domain. □

Claim 3. The family $\{L^n f\}_{n\in\mathbb{N}_0}$ is locally bounded on Ω.

Proof of Claim 3: By Claim 2, a_m is holomorphic on the connected open set Ω. Since it is not identically zero, the zero set

$$Z = \{z \in \Omega \mid a_m(z) = 0\}$$

has no accumulation point in Ω and is closed in Ω.

First, we prove local boundedness at every point of $\Omega \setminus Z$. By Claim 1, we may fix $x_* \in I \setminus Z$ and $M_* > 0$ satisfying the bounds in that claim.

The domain $\Omega \setminus Z$ is path connected. Indeed, two points can be joined by a polygonal path in Ω. Its compact image meets only finitely many zeros, and small detours around those zeros give a path in $\Omega \setminus Z$.

Fix $z \in \Omega \setminus Z$ and a continuous path from x_* to z in this domain. Its image Γ is compact. Choose $R > 0$ small enough that the compact set

$$E = \bigcup_{w\in\Gamma} \overline{D(w, R)}$$

is contained in $\Omega \setminus Z$. To apply Lemma 3.1, set

$$c = \frac{1}{a_m}, \quad b_j = -\frac{a_j}{a_m}, \quad v_{n,j} = (L^n f)^{(j)} \quad (0 \le j < m).$$

For every $n \in \mathbb{N}_0$ and $0 \le j < m - 1$, the definition gives

$$v'_{n,j} = \left((L^n f)^{(j)}\right)' = (L^n f)^{(j+1)} = v_{n,j+1}.$$

Also, $L(L^n f) = L^{n+1} f$ on Ω, so on $\Omega \setminus Z$,

$$\begin{aligned} v'_{n,m-1} &= (L^n f)^{(m)} \\ &= \frac{L^{n+1} f}{a_m} - \sum_{j=0}^{m-1} \frac{a_j}{a_m} (L^n f)^{(j)} \\ &= c v_{n+1,0} + \sum_{j=0}^{m-1} b_j v_{n,j}. \end{aligned}$$

Thus the derivative identities required by Lemma 3.1 hold. By Claim 2, all $v_{n,j}$ are holomorphic on Ω. Since a_m has no zeros on $\Omega \setminus Z$, the coefficients $c, b_0, \ldots, b_{m-1}$ are holomorphic there and hence bounded on the compact set E. Set

$$C_0 = \max\left\{1, \|c\|_E + \sum_{j=0}^{m-1} \|b_j\|_E\right\}, \quad A = \max\left\{\frac{2}{R}, 2C_0\right\}.$$

These choices satisfy Lemma 3.1 on every disk of radius R centered on the path.

Uniform continuity of the path allows a finite subdivision whose consecutive centers have distance less than $\frac{1}{2A}$. Starting with the bound M_* at x_*, each application gives a bound on the next disk at most twice the preceding bound. After finitely many steps, all iterates $L^n f$ are uniformly bounded on a neighborhood of z. Thus the family $\{L^n f\}_{n \in \mathbb{N}_0}$ is locally bounded on $\Omega \setminus Z$.

Next, we prove local boundedness at every point of Z. Fix $z_0 \in Z$. Choose $\rho > 0$ such that $\overline{D(z_0, \rho)} \subseteq \Omega$ and this disk contains no other zero. Its boundary circle is a compact subset of $\Omega \setminus Z$. By the first step, every point of this boundary circle has a neighborhood on which all $L^n f$ are bounded by a common constant independent of n. Compactness gives a finite cover by these neighborhoods, hence a constant C such that

$$|(L^n f)(z)| \le C \quad (n \in \mathbb{N}_0, |z - z_0| = \rho).$$

By Claim 2, every $L^n f$ is holomorphic on a neighborhood of the closed disk, including at z_0. The maximum-modulus principle gives

$$|(L^n f)(z)| \le C \quad (n \in \mathbb{N}_0, |z - z_0| \le \rho).$$

This bounds all $L^n f$ on the neighborhood $D(z_0, \rho)$ of z_0, so the family is locally bounded at z_0. Since $z_0 \in Z$ was arbitrary, the family is locally bounded at every point of Z. Together with the first step, this proves local boundedness on all of Ω. □

Fix a nonempty compact interval $K \subseteq I$. If $\Omega \neq \mathbb{C}$, then K and the closed set $\mathbb{C} \setminus \Omega$ are disjoint, so their distance is positive. Choose $r > 0$ smaller than that distance. If $\Omega = \mathbb{C}$, choose any $r > 0$.

The union

$$K_r = \bigcup_{x \in K} \overline{D(x, r)}$$

is contained in Ω by the choice of r. It is compact because it is the image of $K \times \overline{D(0, r)}$ under addition. By Claim 3 and a finite subcover of K_r, there is a constant $C_K > 0$ bounding all the iterates $L^n f$ on K_r.

For each $x \in K$, the closed disk $\overline{D(x, r)}$ lies in K_r. By Claim 2 and Cauchy's derivative estimate, for every $n, q \in \mathbb{N}_0$,

$$\begin{aligned} \left|(L^n f)^{(q)}(x)\right| &\leq \frac{q!}{r^q} \sup_{|z-x|=r} |(L^n f)(z)| \\ &\leq C_K q! r^{-q}. \end{aligned}$$

The right-hand side is independent of x. Taking the supremum over $x \in K$ therefore gives, for every $n, q \in \mathbb{N}_0$,

$$\left\|(L^n f)^{(q)}\right\|_K = \sup_{x \in K} \left|(L^n f)^{(q)}(x)\right| \leq C_K q! r^{-q}.$$

□

3.3 Analytic fixed-point limits

Theorem 3.3 **Analytic fixed-point limits**

Let $m \in \mathbb{N}$ and define

$$L = \sum_{j=0}^{m} a_j D^j,$$

where $a_0, \ldots, a_m$ are real analytic on I and a_m is not identically zero. If f is real analytic on I and $L^n f$ converges pointwise to g, then

1. $Lg = g$
2. g is real analytic

Proof:

1. Since $L^n f \to g$ pointwise, $((L^n f)(x))_{n\in\mathbb{N}_0}$ is bounded for all $x \in I$, so $\sup_{n\in\mathbb{N}_0} |(L^n f)(x)| < \infty$. Hence, Proposition 3.2 applies, and we obtain that for every compact interval $K \subseteq I$, there exist $C, r > 0$ such that

$$\left\|(L^n f)^{(q)}\right\|_K \leq Cq!r^{-q} \quad \text{for all } n, q \in \mathbb{N}_0. \quad (\heartsuit)$$

   Therefore, Lemma 2.2 applies, and we obtain that g is smooth and that $(L^n f)^{(j)} \to g^{(j)}$ locally uniformly on I for all $j \in \mathbb{N}_0$ ($\bullet$). Locally uniform convergence implies pointwise convergence, so

$$(L^n f)^{(j)}(x) \to g^{(j)}(x) \quad \text{for every } x \in I \text{ and every } j \in \mathbb{N}_0. \quad (\blacklozenge)$$

   Fix $x \in I$. Then

$$\lim_{n\to\infty} (L(L^n f))(x) = \lim_{n\to\infty} (L^{n+1} f)(x)$$

$$\implies \lim_{n\to\infty} \sum_{j=0}^{m} a_j(x) \cdot (L^n f)^{(j)}(x) = g(x) \quad (\text{since } (L^{n+1} f)(x) \to g(x))$$

$$\implies \sum_{j=0}^{m} a_j(x) \cdot \lim_{n\to\infty} (L^n f)^{(j)}(x) = g(x)$$

$$\implies \sum_{j=0}^{m} a_j(x) \cdot g^{(j)}(x) = g(x) \quad (\text{by } \blacklozenge)$$

$$\implies (Lg)(x) = g(x).$$

   Since $x \in I$ was arbitrary, $Lg = g$, as desired.

2. Fix $x_0 \in I$ and choose $s > 0$ such that $K = [x_0 - s, x_0 + s] \subseteq I$. Choose $C, r > 0$ as in ($\heartsuit$) for this interval K. By ($\bullet$), $L^n f \to g$ in $C^\infty_{\text{loc}}(I)$. For each $q \in \mathbb{N}_0$, the seminorm $h \mapsto \left\|h^{(q)}\right\|_K$ is continuous in this topology. Thus ($\heartsuit$) gives

$$\left\|g^{(q)}\right\|_K = \lim_{n\to\infty} \left\|(L^n f)^{(q)}\right\|_K \leq Cq!r^{-q} \quad (q \in \mathbb{N}_0).$$

   Fix x with $|x - x_0| < \min\{s, r\}$. Since g is smooth, Taylor's theorem and the bound with $q = N + 1$ give, for every $N \in \mathbb{N}_0$,

$$\left| g(x) - \sum_{q=0}^{N} \frac{g^{(q)}(x_0)}{q!}(x-x_0)^q \right| \leq \frac{\left\| g^{(N+1)} \right\|_K}{(N+1)!}|x-x_0|^{N+1}$$
$$\leq \frac{C(N+1)!r^{-(N+1)}}{(N+1)!}|x-x_0|^{N+1}$$
$$= C\left(\frac{|x-x_0|}{r}\right)^{N+1}.$$

Since $\frac{|x-x_0|}{r} < 1$, the right side tends to zero as $N \to \infty$. Therefore,

$$g(x) = \sum_{q=0}^{\infty} \frac{g^{(q)}(x_0)}{q!}(x-x_0)^q \quad \text{whenever } |x-x_0| < \min\{s, r\}.$$

Thus g equals its Taylor series on a neighborhood of x_0, so it is real analytic at x_0. Since $x_0 \in I$ was arbitrary, g is real analytic on I, as desired.

□

Remark 3.4

For analytic input, Theorem 3.3 applies to D^r for every $r \in \mathbb{N}$, and to xD and x^2D, including on intervals containing zero. It gives the corresponding fixed-point equations $g^{(r)} = g$, $xg' = g$, and $x^2g' = g$. Our previous results (Edmonstone and Tchouikine 2026, Section 3) allow smooth input for xD and x^2D.

4 A singular counterexample

4.1 Smoothness of the input

Lemma 4.1 **The flat exponential**

Define

$$\eta(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then $\eta \in C^\infty(\mathbb{R})$ and $\eta^{(j)}(0) = 0$ for every $j \in \mathbb{N}_0$.

Proof: First, for every $N \in \mathbb{N}_0$,

$$|x|^{-N} e^{-\frac{1}{x^2}} \longrightarrow 0 \quad \text{as } x \longrightarrow 0.$$

To verify this, put $y = \frac{1}{x^2}$ and choose $q \in \mathbb{N}$ with $q > \frac{N}{2}$. Since $e^y \geq \frac{y^q}{q!}$ for $y > 0$, we have $0 \leq y^{\frac{N}{2}} e^{-y} \leq q! y^{\frac{N}{2}-q} \underset{y\to\infty}{\longrightarrow} 0$.

Off zero, repeated differentiation gives $\eta^{(j)}(x) = P_j\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$ for polynomials P_j. Indeed, starting with $P_0 = 1$, the chain and product rules give

$$P_{j+1}(y) = -y^2 P_j{}'(y) + 2y^3 P_j(y).$$

Each of these expressions extends continuously by zero at the origin, by the preceding decay estimate. Its difference quotient at zero is $\frac{P_j\left(\frac{1}{x}\right)}{x} e^{-\frac{1}{x^2}}$, which also tends to zero. Inductively, each continuous extension is differentiable at zero with derivative zero, and its derivative is the next continuous extension. This proves smoothness and all the stated derivative values. □

Set $f = 1 - \eta$. Then $f \in C^\infty(\mathbb{R})$, $f(0) = 1$, and $f^{(j)}(0) = 0$ for $j \in \mathbb{N}$. Its Taylor series at zero is the constant one. Since $f(x) < 1$ for every $x \neq 0$, it cannot agree with that Taylor series on any neighborhood of zero. Thus f is not real analytic at zero.

4.2 Construction of the operator

Lemma 4.2 **Change of variable and integrating factor**

Define

$$p(t) := te^{-t} \quad (t > 0), \quad t(x) := \frac{1}{x^2} \quad (x \neq 0),$$

and let

$$L := -\frac{x^3}{2} D + (1 - x^2).$$

For each $H \in C^\infty((0, \infty))$, define

$$w_H(x) := p(t(x)) H(t(x)) \quad (x \in \mathbb{R} \setminus \{0\}).$$

Then

$$L w_H = w_{H'} \quad \text{on } \mathbb{R} \setminus \{0\}.$$

Proof: Fix $H \in C^\infty((0, \infty))$ and $x \neq 0$, and write $t = t(x)$. Since

$$t'(x) = -\frac{2}{x^3}, \quad -\frac{x^3}{2}t'(x) = 1,$$

the chain rule gives

$$\begin{aligned}(Lw_H)(x) &= \frac{\mathrm{d}}{\mathrm{d}t}[p(t)H(t)] + (1-x^2)p(t)H(t) \\ &= p(t)H'(t) + \left[p'(t) + \left(1 - \frac{1}{t}\right)p(t)\right]H(t).\end{aligned}$$

Here we used $x^2 = \frac{1}{t}$. Since $p(t) = te^{-t}$,

$$p'(t) + \left(1 - \frac{1}{t}\right)p(t) = (1-t)e^{-t} + (t-1)e^{-t} = 0.$$

Consequently,

$$(Lw_H)(x) = p(t(x))H'(t(x)) = w_{H'}(x).$$

Since $x \neq 0$ was arbitrary, the identity holds on $\mathbb{R} \setminus \{0\}$. □

For the counterexample, we use the sequence

$$H_n(t) := \int_0^1 s^n e^{st}\,\mathrm{d}s \quad (t > 0, n \in \mathbb{N}_0).$$

Differentiating under the integral gives $H_n' = H_{n+1}$. Repeated differentiation is justified on each compact t-interval by continuity and uniform boundedness of the integrand and its t-derivatives on the rectangle. In particular, every H_n is smooth on $(0, \infty)$. Also,

$$H_0(t) = \frac{e^t - 1}{t}, \quad p(t)H_0(t) = 1 - e^{-t}.$$

Thus

$$w_{H_0}(x) = 1 - e^{-\frac{1}{x^2}} \quad (x \neq 0),$$

which extends smoothly to the origin with value one by Lemma 4.1. Both coefficients of L are polynomials, and its leading coefficient has exactly one zero, of order three, at the origin.

4.3 Orbit formula and convergence

Theorem 4.3 **A discontinuous limit with polynomial coefficients**

For $h \in C^\infty(\mathbb{R})$, define

$$Lh = -\frac{x^3}{2}h' + (1-x^2)h, \quad f = 1 - \eta,$$

where η is the function in Lemma 4.1. Then every $L^n f$ is smooth, and for every $n \in \mathbb{N}_0$,

$$(L^n f)(x) = \begin{cases} \frac{1}{x^2}\int_0^1 s^n e^{-\frac{1-s}{x^2}}\,\mathrm{d}s & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Moreover, $0 \leq L^{n+1}f \leq L^n f \leq 1$, and the sequence converges pointwise to the discontinuous function

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Proof: Smoothness of f follows from Lemma 4.1. Since differentiation and multiplication by polynomial coefficients preserve smoothness, induction shows that every $L^n f$ is smooth on all of $\mathbb{R}$.

On $\mathbb{R} \setminus \{0\}$, we have $f = w_{H_0}$. By Lemma 4.2 and the identity $H_n{}' = H_{n+1}$, we have $Lw_{H_n} = w_{H_{n+1}}$. Induction therefore gives, with $t = \frac{1}{x^2}$,

$$(L^n f)(x) = p(t)H_n(t) = t\int_0^1 s^n e^{-(1-s)t}\,\mathrm{d}s.$$

Substituting $t = \frac{1}{x^2}$ gives the asserted formula.

At the origin, every smooth h satisfies

$$(Lh)(0) = h(0).$$

Since $f(0) = 1$, induction gives $(L^n f)(0) = 1$ for all n.

For a fixed $x \neq 0$, the integral formula implies

$$0 \leq (L^n f)(x) \leq \frac{1}{x^2}\int_0^1 s^n\,\mathrm{d}s = \frac{1}{x^2(n+1)}.$$

The right side tends to zero as n tends to infinity. Together with the value at zero, this proves the pointwise limit.

Since $0 \le s^{n+1} \le s^n$ on $[0,1]$, the same formula shows $0 \le (L^{n+1}f)(x) \le (L^n f)(x)$ off zero; the inequalities also hold at zero. Finally, $f \le 1$. Hence the orbit is nonnegative, decreasing, and bounded by one everywhere. □

4.4 Failure of local uniform convergence

The convergence is uniform on every compact set avoiding zero: if $|x| \ge \delta > 0$, the bound in the proof is at most $\frac{1}{\delta^2(n+1)}$. On any interval containing zero in its interior, however, convergence cannot be uniform. In fact, each $L^n f$ is continuous and equals one at zero, while g is zero at every nearby nonzero point. Consequently

$$\sup_{0<|x|\le\varepsilon} |(L^n f)(x) - g(x)| = 1 \quad (\varepsilon > 0).$$

There is also an explicit failure of uniform derivative bounds. Integrating the formula for $L^n f$ by parts in s, for $n \in \mathbb{N}$, gives

$$(L^n f)(x) = 1 - nx^2 (L^{n-1} f)(x) \quad (x \neq 0).$$

Indeed, the differential of $e^{-(1-s)t}$ is $te^{-(1-s)t}\,\mathrm{d}s$; the boundary term is one and the remaining integral is $\frac{n}{t} L^{n-1} f$. By continuity the recurrence holds at zero as well. Differentiating it twice at zero therefore gives

$$(L^n f)^{(2)}(0) = -2n(L^{n-1} f)(0) = -2n.$$

Consequently, the local bound on second derivatives required in Lemma 2.2 fails at the origin.

For this same operator, Theorem 3.3 guarantees an analytic fixed-point limit whenever an analytic input has a pointwise convergent orbit. The failure here occurs for the smooth input $f = 1 - \eta$, whose nonanalyticity at zero was established above.

4.5 The positive-order assumption

Positive order is also necessary. On $I = (-1, 1)$, let

$$Lh = (1 - x^2)h, \quad f = 1.$$

Both the coefficient and the input are real analytic, but

$$(L^n f)(x) = (1 - x^2)^n \longrightarrow \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$

Indeed, $0 < 1 - x^2 < 1$ whenever $0 < |x| < 1$, whereas the value at zero is always one. Thus Theorem 3.3 does not extend to operators of order zero.

References

Edmonstone, Jake, and Matthew Tchouikine. 2026. *Iteration of Linear Differential Operators*. Directed Research Program report. University of Waterloo.

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