Iteration of linear differential operators

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1. The original problem

Does iterating the derivative infinitely many times give a smooth function whenever it converges?

Asked 4 years, 6 months ago Modified 3 years, 6 months ago Viewed 5k times

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I am a graduate student and I've been thinking about this fun but frustrating problem for some time. Let $d = \frac{d}{dx}$, and let $f \in C^\infty(\mathbb{R})$ be such that for every real x,

$$g(x) := \lim_{n\to\infty} d^n f(x)$$

converges. A simple example for such an f would be $ce^x + h(x)$ for any constant c where $h(x)$ converges to 0 everywhere under this iteration (in fact my hunch is that every such f is of this form), eg. $h(x) = e^{x/2}$ or simply a polynomial, of course.

I've been trying to show that g is, in fact, differentiable, and thus is a fixed point of d. Whether this is true would provide many interesting properties from a dynamical systems point of view if one can generalize to arbitrary smooth linear differential operators, although they might be too good to be true.

Perhaps this is a known result? If so I would greatly appreciate a referer
has a trivial counterexample I've missed, please let me know. Otherwise
with some tricky double limit using tricks such as in this MSE answer, to

asked Jan 5, 2022 at 6:05
Paul Cusson

Any help is kindly appreciated.

1.1 The original question

Let $f \in C^\infty(\mathbb{R})$, and write

$$L_1 := \frac{\mathrm{d}}{\mathrm{d}x}.$$

Suppose that for every $x \in \mathbb{R}$ the pointwise limit

$$g(x) := \lim_{n\to\infty} L_1^n f(x)$$

exists.

Question. Must g be smooth? Moreover, can we prove $L_1 g = g$?

A direct answer:

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I was able to adapt the accepted answer to this MathOverflow post to positively answer the question. The point is that one can squeeze more out of Petrov's Baire category argument if one applies it to the "singular set" of the function, rather than to an interval.

The key step is to establish

> **Theorem 1**. Let $f \in C^\infty(\mathbf{R})$ be such that the quantity $M(x) := \sup_{m\geq 0} |f^{(m)}(x)|$ is finite for all x. Then f is the restriction to $\mathbf{R}$ of an entire function (or equivalently, f is real analytic with an infinite radius of convergence).

The convergence hypothesis actually forces f to be a restriction of an entire function.

answered Jan 6, 2022 at 4:06

Terry Tao

1.2 Tao's analyticity theorem

Theorem 1

Let $f \in C^\infty(\mathbb{R})$, and suppose that for every $x \in \mathbb{R}$,

$$M(x) := \sup_{m \geq 0} \left| f^{(m)}(x) \right| < \infty.$$

Then f is the restriction to $\mathbb{R}$ of an entire function. Equivalently, f is real analytic with infinite radius of convergence.

1.3 Why Tao's theorem applies

For each fixed $x \in \mathbb{R}$, the sequence

$$\left(f^{(n)}(x)\right)_{n\geq 0}$$

converges, hence is bounded. Therefore

$$M(x) = \sup_{n\geq 0}\left|f^{(n)}(x)\right| < \infty,$$

so Tao's theorem applies immediately: f is the restriction of an entire function.

1.4 One bound for every derivative

Since f is the restriction of an entire function, expand $f^{(k)}$ at 0. Every coefficient is controlled by the *single* number $M(0)$:

$$f^{(k)}(x) = \sum_{n=0}^{\infty} \frac{f^{(n+k)}(0)}{n!} x^n,$$

$$\implies \left|f^{(k)}(x)\right| \leq \sum_{n=0}^{\infty} \frac{M(0)}{n!} |x|^n = M(0)e^{|x|}.$$

Note that for all $n \in \mathbb{N}$ and $x \in [a, b]$,

$$\left|f^{(n+1)}(x)\right| \leq C_{[a,b]} := \sup_{x \in [a,b]} M(0)e^{|x|} < \infty.$$

1.5 Pass the limit through the integral

Fix $a < b$. For every n, the Fundamental Theorem of Calculus gives the identity

$$f^{(n)}(b) - f^{(n)}(a) = \int_a^b f^{(n+1)}(x)\,\mathrm{d}x$$

For $x \in [a, b]$, we have

$$\lim_{n\to\infty} f^{(n+1)}(x) = g(x) \quad \text{and} \quad \left|f^{(n+1)}(x)\right| \leq C_{[a,b]}.$$

Therefore, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}\lim_{n\to\infty} \int_a^b f^{(n+1)}(x)\,\mathrm{d}x &= \int_a^b \lim_{n\to\infty} f^{(n+1)}(x)\,\mathrm{d}x \\ &= \int_a^b g(x)\,\mathrm{d}x.\end{aligned}$$

1.6 The limit is continuous

Using the locally uniform bound from before, we show that g is continuous,

$$g(b) - g(a) = \int_a^b g(x)\,\mathrm{d}x,$$

Therefore,

$$\begin{aligned} |g(b) - g(a)| &\leq \int_a^b |g(x)|\,\mathrm{d}x \\ &\leq C_{[a,b]}(b-a). \end{aligned}$$

1.7 The limit satisfies $L_1 g = g$

Fix b. The integral identity becomes

$$g(x) = g(b) + \int_b^x g(t)\,\mathrm{d}t.$$

Since g is continuous, the Fundamental Theorem of Calculus yields

$$g'(x) = g(x).$$

Thus $L_1 g = g$, so g is smooth. Solving the ODE gives

$$g(x) = Ce^x$$

for some $C \in \mathbb{R}$.

2. Extensions

2.1 $L_2 f := a f'$

2.1.1 The fixed-point equation

Let $a \in C^\infty(\mathbb{R})$ be nowhere zero, and define

$$L_2 f := af'.$$

Suppose that the pointwise limit

$$g(x) := \lim_{n\to\infty} L_2^n f(x)$$

exists for every $x \in \mathbb{R}$.

We will show that

$$L_2 g = g.$$

2.1.2 Change coordinates

Introduce the new coordinate

$$\Phi : \mathbb{R} \to \mathbb{R}, x \mapsto \int_0^x \frac{1}{a(t)} \, \mathrm{d}t.$$

Since a is nowhere zero, Φ is a diffeomorphism from $\mathbb{R}$ onto an open interval $I := \Phi(\mathbb{R})$. Set

$$F := f \circ \Phi^{-1} : I \to \mathbb{R}.$$

The chain rule gives

$$F' = (L_2 f) \circ \Phi^{-1},$$

and therefore, for every $n \geq 0$,

$$F^{(n)} = (L_2^n f) \circ \Phi^{-1}.$$

2.1.3 Pull back the ordinary result

Note that

$$\begin{aligned}\lim_{n\to\infty} F^{(n)}(y) &= \lim_{n\to\infty} (L_2^n f)\big(\Phi^{-1}(y)\big) \\ &= g\big(\Phi^{-1}(y)\big)\end{aligned}$$

exists. So, we can define $G(y) := \lim_{n\to\infty} F^{(n)}(y)$. The ordinary derivative result on the interval I gives

$$G' = G.$$

2.1.4 Recover the fixed-point equation

Since $G = g \circ \Phi^{-1}$, then $g = G \circ \Phi$. Since $\Phi' = \frac{1}{a}$ (by FTC), then

$$\begin{aligned} g' &= (G' \circ \Phi) \cdot \Phi' \\ &= (G \circ \Phi) \cdot \frac{1}{a} \\ &= g \cdot \frac{1}{a}. \end{aligned}$$

Hence

$$\begin{aligned} L_2 g &= a g' \\ &= a\left(g \cdot \frac{1}{a}\right) \\ &= g. \end{aligned}$$

2.2 $L_3 f := af' + bf$

2.2.1 Absorbing the zeroth-order term

Let $a, b \in C^\infty(\mathbb{R})$, with a nowhere zero, and suppose $L_3^n f \to g$ pointwise.

Use the same coordinate change as for af', together with an integrating factor, to define

$$\Phi(x) := \int_0^x \frac{1}{a(t)}\,\mathrm{d}t, \quad \beta := b \circ \Phi^{-1}.$$

$$w(y) := \exp\left(\int_0^y \beta(s)\,\mathrm{d}s\right), \quad (Tf)(y) := w(y) f(\Phi^{-1}(y)).$$

The chain rule gives

$$T(L_3 f) = (Tf)', \quad T(L_3^n f) = (Tf)^{(n)}.$$

Hence $(Tf)^{(n)} \to Tg$. By the ordinary derivative result, $(Tg)' = Tg$. But $(Tg)' = T(L_3 g)$, and T is injective, so $\boldsymbol{L_3 g = g}$.

2.3 $L_4 f := x f'$

2.3.1 A simple zero

Suppose $g(x) := \lim_{n\to\infty} L_4^n f(x)$ exists pointwise.

Change coordinates. For $\sigma \in \{-1, 1\}$, set

$$\Phi_\sigma(x) := \log(\sigma x), \quad F_\sigma := f \circ \Phi_\sigma^{-1}.$$

$$F_\sigma^{(n)} = (L_4^n f) \circ \Phi_\sigma^{-1}.$$

The ordinary derivative result makes F_σ entire.

Remove the affine part. If $a := f(0)$ and $b := f'(0)$, define

$$H_\sigma(y) := F_\sigma(y) - a - b\Phi_\sigma^{-1}(y).$$

Apply Phragmén–Lindelöf. It forces $H_\sigma = 0$. Hence $f(x) = a + bx$ on $\mathbb{R}$, so $g(x) = bx$ and $L_4 g = g$.

2.4 $L_5 f := x^2 f'$

2.4.1 A second-order zero

Suppose $g(x) := \lim_{n\to\infty} L_5^n f(x)$ exists pointwise.

Change coordinates. On each half-line, $\Phi(x) := -\frac{1}{x}$ turns L_5 into ordinary differentiation, so

$$g(x) = C_+ e^{-\frac{1}{x}} \quad (x > 0), \quad g(x) = C\_e^{-\frac{1}{x}} \quad (x < 0).$$

At the origin. As $x \to 0^-$, $e^{-\frac{1}{x}} \to \infty$, so smoothness forces $C\_ = 0$. Also, $(L_5 h)(0) = 0$ for every smooth h, hence $g(0) = 0$.

Conclusion.

$$g(x) = \begin{cases} Ce^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

The right-hand term vanishes at 0. Therefore $g \in C^\infty(\mathbb{R})$ and $L_5 g = g$.

3. Future directions

3.1 Open problems beyond first-order ODEs

The guiding question is:

$$L^n f \to g \text{ pointwise} \quad \Longrightarrow \quad g \in C^\infty(\mathbb{R}) \text{ and } Lg = g?$$

1. Higher-order ODEs. Study

$$Lf(x) = \sum_{k=0}^{m} a_k(x) f^{(k)}(x), \quad m \geq 2.$$

What do we do when the first-order change of coordinates is unavailable?

2. Partial differential operators.

$$Lf(x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha f(x) \quad \text{on} \quad \mathbb{R}^d,$$

Questions?