Iteration of Linear Differential Operators

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1.1 The original problem

Problem **Iteration of the derivative**

Let $d = \frac{\mathrm{d}}{\mathrm{d}x}$, and let $f \in C^\infty(\mathbb{R})$. Suppose that, for every $x \in \mathbb{R}$, the limit

$$g(x) := \lim_{n\to\infty} d^n f(x) = \lim_{n\to\infty} f^{(n)}(x)$$

exists. We want to show that g is differentiable and, in fact, satisfies $g' = g$.

1.2 Analyticity from pointwise derivative bounds

Theorem 1.1 **Analyticity theorem**

Let $f \in C^\infty(\mathbb{R})$. Define

$$M(x) := \sup\left\{\left|f^{(k)}(x)\right| \;\middle|\; k \geq 0\right\},$$

and assume that $M(x) < \infty$ for every $x \in \mathbb{R}$. Then f is the restriction to $\mathbb{R}$ of an entire function.

Proof: Suppose this is not the case. Let X denote the set of real numbers x for which there does not exist any entire function that agrees with f on a neighbourhood of x. If $X = \varnothing$, then every point has a neighbourhood on which f agrees with some entire function. These entire functions agree on overlaps by analytic continuation, and hence glue to a single entire function agreeing with f on all of $\mathbb{R}$, contradicting our assumption. Therefore X is non-empty.

Claim 1: The set X is closed.

Verification of Claim 1: Let $(x_n) \subseteq X$. Suppose $x_n \to x \notin X$. Then there exists a neighbourhood $U \subseteq \mathbb{R}$ of x and an entire function agreeing with f on U. Since $x_n \to x$, we have $x_N \in U$ for some $N \in \mathbb{N}$. Thus U is a neighbourhood of x_N on which an entire function agrees with f, contradicting $x_N \in X$. Therefore $x \in X$, so X is closed.

Now define $S_n := \{x \in \mathbb{R} \mid M(x) \leq n\}$. Then,

$$\begin{aligned} S_n &= \left\{x \in \mathbb{R} \;\middle|\; f^{(k)}(x) \in [-n, n] \;\forall k \geq 0\right\} \\ &= \bigcap_{k\geq 0} \left(\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right)^{-1}([-n, n]) \end{aligned}$$

Hence each S_n is closed. Also note that

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} S_n$$

So

$$X = \bigcup_{n\in\mathbb{N}} (S_n \cap X)$$

Since X is closed in $\mathbb{R}$, it is complete in the induced metric. Since X is also non-empty, BCT implies that there exists an $N \in \mathbb{N}$ with $S_N \cap X$ having non-empty interior relative to X. In other words, there exist $a, b \in \mathbb{R}$ with $a < b$ such that

$$\varnothing \neq (a, b) \cap X \subseteq S_N \cap X$$

If $(a, b) \setminus X = \varnothing$, then $(a, b) \subseteq X \subseteq S_N$. Hence

$$\left|f^{(m)}(x)\right| \leq N$$

for all $x \in (a, b)$ and all $m \geq 0$. The Taylor argument below then shows that every $x_0 \in (a, b)$ has a neighbourhood on which f agrees with an entire function, contradicting $(a, b) \subseteq X$. Thus $(a, b) \setminus X$ is non-empty.

Now let (c, e) be a maximal interval in the open set $(a, b) \setminus X$. Since $(c, e) \subseteq (a, b) \setminus X$, for every $x \in (c, e)$ there is some neighbourhood of x on which f agrees with an entire function. By analytic continuation, these entire functions agree on overlaps, and hence f agrees with a single entire function F on all of (c, e).

Claim 2: At least one endpoint of (c, e) lies in $(a, b) \cap X$.

Verification of Claim 2: Suppose first that $c \in (a, b)$. If $c \notin X$, then $c \in (a, b) \setminus X$. Since $(a, b) \setminus X$ is open, some neighbourhood of c is contained in $(a, b) \setminus X$. This would extend (c, e) to the left, contradicting maximality. Hence, if $c \in (a, b)$, then $c \in X$. Similarly, if $e \in (a, b)$, then $e \in X$.

If neither endpoint lies in (a, b), then $(c, e) = (a, b)$, contradicting $(a, b) \cap X \neq \varnothing$. Therefore at least one endpoint lies in $(a, b) \cap X$.

The argument below treats the case $c \in (a, b) \cap X$. If instead $e \in (a, b) \cap X$, the same argument works with e in place of c, expanding around e and using $x \to c^+$ at the end. Since $(a, b) \cap X \subseteq S_N$, we have $c \in S_N$, and so

$$\left|f^{(m)}(c)\right| \leq N$$

for all $m \geq 0$.

Since $F|_{(c,e)} = f|_{(c,e)}$ and both functions are C^∞ there, we have $F^{(r)}|_{(c,e)} = f^{(r)}|_{(c,e)}$ for every $r \geq 0$. Therefore, using continuity of $F^{(r)}$ and $f^{(r)}$,

$$F^{(r)}(c) = \lim_{x \to c^+} F^{(r)}(x) = \lim_{x \to c^+} f^{(r)}(x) = f^{(r)}(c)$$

Now let $x \in (c, e)$. Since F is entire, Taylor expansion of $F^{(m)}$ around c gives, for each $m \geq 0$,

$$f^{(m)}(x) = F^{(m)}(x) = \sum_{j=0}^{\infty} \frac{F^{(m+j)}(c)}{j!}(x-c)^j = \sum_{j=0}^{\infty} \frac{f^{(m+j)}(c)}{j!}(x-c)^j$$

Thus,

$$\begin{aligned}
\left|f^{(m)}(x)\right| &\leq \sum_{j=0}^{\infty} \frac{\left|f^{(m+j)}(c)\right|}{j!}|x-c|^j \\
&\leq \sum_{j=0}^{\infty} \frac{N}{j!}|x-c|^j \\
&\leq \sum_{j=0}^{\infty} \frac{N}{j!}(b-a)^j \\
&= N \exp(b-a)
\end{aligned}$$

for all $m \geq 0$ and all $x \in (c, e)$. Since $f \in C^\infty(\mathbb{R})$, each $f^{(m)}$ is continuous, so letting $x \to e^-$ gives

$$\left|f^{(m)}(e)\right| \leq N \exp(b-a)$$

for all $m \geq 0$. Hence the bound holds for all $m \geq 0$ and all $x \in [c, e]$.

Now let (c, e) vary over all maximal intervals in $(a, b) \setminus X$. If $x \in (a, b) \setminus X$, then x lies in one such interval, and the above argument gives

$$\left|f^{(m)}(x)\right| \leq N \exp(b-a)$$

for all $m \geq 0$. On the other hand, if $x \in (a, b) \cap X$, then $x \in S_N$, and so

$$\left|f^{(m)}(x)\right| \leq N \leq N \exp(b-a)$$

for all $m \geq 0$. Hence

$$\left|f^{(m)}(x)\right| \leq N \exp(b-a)$$

for all $x \in (a, b)$ and all $m \geq 0$.

We now show that f is real analytic on (a, b). Let $x_0 \in (a, b)$, and choose $r > 0$ such that $(x_0 - r, x_0 + r) \subseteq (a, b)$. For $x \in (x_0 - r, x_0 + r)$ and $K \geq 0$, Taylor's theorem with remainder gives

$$f(x) = \sum_{j=0}^{K} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{f^{(K+1)}(\xi)}{(K+1)!}(x - x_0)^{K+1}$$

for some ξ between x and x_0. Therefore,

$$\left| f(x) - \sum_{j=0}^{K} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \right| \leq \frac{N \exp(b-a)}{(K+1)!} |x - x_0|^{K+1}$$

Letting $K \to \infty$, the right hand side goes to 0. Hence f agrees with its Taylor series on $(x_0 - r, x_0 + r)$. Since $x_0 \in (a, b)$ was arbitrary, f is real analytic on (a, b).

Moreover, the same bound shows that this Taylor series has infinite radius of convergence. Indeed, for every $z \in \mathbb{C}$,

$$\sum_{j=0}^{\infty} \left| \frac{f^{(j)}(x_0)}{j!}(z - x_0)^j \right| \leq \sum_{j=0}^{\infty} \frac{N \exp(b-a)}{j!} |z - x_0|^j = N \exp(b-a) \exp(|z - x_0|) < \infty$$

Thus the Taylor series at x_0 defines an entire function which agrees with f on a neighbourhood of x_0. Hence $x_0 \notin X$. Since $x_0 \in (a, b)$ was arbitrary, $(a, b) \cap X = \varnothing$. This contradicts $(a, b) \cap X \neq \varnothing$.

Therefore our supposition was false, so $X = \varnothing$. Hence every point has a neighbourhood on which f agrees with an entire function. By analytic continuation, these local entire functions agree on overlaps and glue to a single entire function whose restriction to $\mathbb{R}$ is f. Therefore f is real analytic with infinite radius of convergence. □

1.3 Solution of the original problem

Theorem 1.2 **Pointwise limits of iterated derivatives**

Let $f \in C^\infty(\mathbb{R})$ and suppose that

$$g(x) := \lim_{n \to \infty} f^{(n)}(x)$$

exists for every $x \in \mathbb{R}$. Then $g \in C^\infty(\mathbb{R})$ and $g' = g$.

Proof: Assume that $f \in C^\infty(\mathbb{R})$ and that

$$g(x) := \lim_{n\to\infty} f^{(n)}(x)$$

exists for every $x \in \mathbb{R}$. Then for each fixed x, the sequence $\left(f^{(n)}(x)\right)_{n\geq 0}$ is convergent, hence bounded. Therefore the hypotheses of Theorem 1.1 are satisfied, so f is the restriction of an entire function.

Indeed, for each $x \in \mathbb{R}$, set

$$M(x) := \sup_{n\geq 0}\left|f^{(n)}(x)\right|.$$

This quantity is finite because $\left(f^{(n)}(x)\right)_{n\geq 0}$ converges.

For all $k \geq 0$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} \left|f^{(k)}(x)\right| &= \left|\sum_{n=0}^{\infty} \frac{f^{(n+k)}(0)}{n!} x^n\right| \\ &\leq \sum_{n=0}^{\infty} \frac{\left|f^{(n+k)}(0)\right|}{n!} |x|^n \\ &\leq \sum_{n=0}^{\infty} \frac{M(0)}{n!} |x|^n \\ &= M(0)\exp(|x|). \end{aligned} \tag{1.1}$$

Let $a < b$. By the Fundamental Theorem of Calculus, we have

$$f^{(n)}(b) - f^{(n)}(a) = \int_a^b f^{(n+1)}(x)\,\mathrm{d}x.$$

By (1.1), we may apply Lebesgue's Dominated Convergence Theorem as $n \to \infty$ to obtain

$$g(b) - g(a) = \int_a^b g(x)\,\mathrm{d}x,$$

and hence

$$\begin{aligned} |g(b) - g(a)| &\leq \int_a^b |g(x)|\,\mathrm{d}x \\ &\leq M(0)\exp(\max(|a|, |b|))(b - a). \end{aligned}$$

Thus g is locally Lipschitz and, in particular, continuous. Fixing a and differentiating with respect to b, the Fundamental Theorem of Calculus gives

$$g'(b) = g(b).$$

Therefore $g' = g$. Since g is continuous, this identity can be differentiated inductively to conclude that g is smooth. □

Lemma 1.3 **Local form of the ordinary derivative result**

Let $I \subseteq \mathbb{R}$ be a non-empty open interval and let $F \in C^\infty(I)$. If $\lim_{n\to\infty} F^{(n)}(y)$ exists for every $y \in I$, then F is the restriction to I of an entire function. Moreover, the pointwise limit G satisfies $G' = G$ on I.

Proof: To see this, apply the Baire argument and Taylor estimate from Theorem 1.1 on compact subintervals of I. Thus every point of I has a neighbourhood on which F agrees with an entire function. These local entire functions agree on overlaps by analytic continuation, and hence glue to a single entire function agreeing with F on all of I. The dominated-convergence step is then applied on an arbitrary compact interval $[a, b] \subseteq I$, giving

$$G(a) - G(b) = \int_b^a G(x)\,\mathrm{d}x.$$

Since $[a, b]$ was arbitrary, $G' = G$ on all of I. □

2 First-order linear differential operators

2.1 The operator $Lf = af'$

Theorem 2.1 **Nowhere-vanishing leading coefficient**

Let $a \in C^\infty(\mathbb{R})$ be nowhere zero, and define the operator

$$Lf := af'$$

on $C^\infty(\mathbb{R})$. Suppose that for every $x \in \mathbb{R}$, the limit

$$g(x) := \lim_{n\to\infty} L^n f(x)$$

exists. We want to show that g is differentiable, and in fact satisfies

$$Lg = g,$$

or equivalently,

$$ag' = g.$$

Proof: Define

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(x) := \int_0^x \frac{1}{a(t)}\,\mathrm{d}t.$$

Since a is nowhere zero, it has constant sign. Thus $\Phi'(x) = \frac{1}{a(x)}$ is nowhere zero, so Φ is a smooth diffeomorphism from $\mathbb{R}$ onto the open interval

$$I := \Phi(\mathbb{R}) \subseteq \mathbb{R}.$$

Let $\Phi^{-1} : I \to \mathbb{R}$ denote the inverse diffeomorphism. Define

$$F := f \circ \Phi^{-1} : I \to \mathbb{R}.$$

Then, for $y \in I$, the inverse derivative formula gives

$$\left(\Phi^{-1}\right)'(y) = \frac{1}{\Phi'(\Phi^{-1}(y))} = \frac{1}{\frac{1}{a(\Phi^{-1}(y))}} = a(\Phi^{-1}(y)).$$

Hence, by the chain rule,

$$\begin{aligned} F' &= \left(f' \circ \Phi^{-1}\right)\left(\Phi^{-1}\right)' \\ &= \left(f' \circ \Phi^{-1}\right)\left(a \circ \Phi^{-1}\right) \\ &= (Lf) \circ \Phi^{-1}. \end{aligned}$$

Applying the same computation to Lf gives

$$\begin{aligned} F'' &= \left((Lf)' \circ \Phi^{-1}\right)\left(\Phi^{-1}\right)' \\ &= \left((Lf)' \circ \Phi^{-1}\right)\left(a \circ \Phi^{-1}\right) \\ &= (L(Lf)) \circ \Phi^{-1} \\ &= \left(L^2 f\right) \circ \Phi^{-1}. \end{aligned}$$

Applying the same identity inductively gives

$$F^{(n)} = (L^n f) \circ \Phi^{-1}$$

on I for every $n \geq 0$.

Now define

$$G : I \to \mathbb{R}, \quad G(y) := \lim_{n\to\infty} F^{(n)}(y).$$

This limit exists because for every $y \in I$, we have $\Phi^{-1}(y) \in \mathbb{R}$, and so

$$G(y) = \lim_{n\to\infty} (L^n f)\left(\Phi^{-1}(y)\right) = g\left(\Phi^{-1}(y)\right).$$

Thus

$$G = g \circ \Phi^{-1}.$$

Now $F \in C^\infty(I)$, and we have just shown that $\lim_{n\to\infty} F^{(n)}(y)$ exists for every $y \in I$. The proof of the original problem applies locally on any open interval, so applying it to $F : I \to \mathbb{R}$ gives

$$G' = G.$$

Since $G = g \circ \Phi^{-1}$, we also have $g = G \circ \Phi$. Therefore, for $x \in \mathbb{R}$,

$$\begin{aligned} g' &= (G' \circ \Phi)\Phi' \\ &= (G \circ \Phi)\frac{1}{a} \\ &= \frac{g}{a}. \end{aligned}$$

Hence

$$ag' = g,$$

or equivalently, $Lg = g$. □

2.2 The operator $Lf = af' + f$

Theorem 2.2 **The case $b = 1$**

Let $a \in C^\infty(\mathbb{R})$ be nowhere zero, and define the operator

$$Lf := af' + f$$

on $C^\infty(\mathbb{R})$. Suppose that for every $x \in \mathbb{R}$, the limit

$$g(x) := \lim_{n\to\infty} L^n f(x)$$

exists. We want to show that g is differentiable, and in fact satisfies

$$Lg = g$$

Moreover, g is constant.

Proof: Define

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(x) := \int_0^x \frac{1}{a(t)}\,\mathrm{d}t,$$

and let

$$I := \Phi(\mathbb{R}) \subseteq \mathbb{R}$$

As before, Φ is a smooth diffeomorphism from $\mathbb{R}$ onto I. Let $\Phi^{-1} : I \to \mathbb{R}$ denote the inverse diffeomorphism. Define

$$E : I \to \mathbb{R}, \quad E(y) := \exp(y)$$

Then $E' = E$. Define the function

$$F := E\left(f \circ \Phi^{-1}\right) : I \to \mathbb{R}$$

Then, using the same calculation as in the first extension,

$$\begin{aligned} F' &= E'\big(f \circ \Phi^{-1}\big) + E\big((af') \circ \Phi^{-1}\big) \\ &= E\big((f + af') \circ \Phi^{-1}\big) \\ &= E\big((Lf) \circ \Phi^{-1}\big) \end{aligned}$$

Applying the same computation to Lf gives

$$\begin{aligned} F'' &= E'\big((Lf) \circ \Phi^{-1}\big) + E\big((a(Lf)') \circ \Phi^{-1}\big) \\ &= E\big(((Lf) + a(Lf)') \circ \Phi^{-1}\big) \\ &= E\big((L(Lf)) \circ \Phi^{-1}\big) \\ &= E\big((L^2 f) \circ \Phi^{-1}\big) \end{aligned}$$

Applying the same identity inductively gives

$$F^{(n)} = E\big((L^n f) \circ \Phi^{-1}\big)$$

on I for every $n \geq 0$.

Now define

$$G : I \to \mathbb{R}, \quad G(y) := \lim_{n \to \infty} F^{(n)}(y)$$

This limit exists because for every $y \in I$,

$$G(y) = E(y) \lim_{n \to \infty} (L^n f)\big(\Phi^{-1}(y)\big) = E(y) g\big(\Phi^{-1}(y)\big)$$

Thus

$$G = E\big(g \circ \Phi^{-1}\big)$$

Now $F \in C^\infty(I)$, and we have just shown that $\lim_{n \to \infty} F^{(n)}(y)$ exists for every $y \in I$. The proof of the original problem applies locally on any open interval, so applying it to $F : I \to \mathbb{R}$ gives

$$G' = G$$

Then

$$G' = E'\big(g \circ \Phi^{-1}\big) + E\big(g \circ \Phi^{-1}\big)' = E\big(g \circ \Phi^{-1}\big) + E\big(g \circ \Phi^{-1}\big)'$$

Since $G' = G = E\big(g \circ \Phi^{-1}\big)$, we get

$$\big(g \circ \Phi^{-1}\big)' = 0$$

Composing with Φ gives

$$g' = 0$$

Hence

$$ag' + g = g,$$

and therefore $Lg = g$.

It is worth noting that in this case the conclusion is stronger than just $Lg = g$: since a is nowhere zero, the equation $ag' + g = g$ forces $g' = 0$. Thus convergence under this operator forces the limiting function g to be constant. □

2.3 The general operator $Lf = af' + bf$

Theorem 2.3 **General nondegenerate first-order operators**

Let $a, b \in C^\infty(\mathbb{R})$, with a nowhere zero, and define the operator

$$Lf := af' + bf$$

on $C^\infty(\mathbb{R})$. Suppose that for every $x \in \mathbb{R}$, the limit

$$g(x) := \lim_{n\to\infty} L^n f(x)$$

exists. We want to show that g is differentiable, and in fact satisfies

$$Lg = g,$$

or equivalently,

$$ag' + bg = g$$

Proof: Define

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(x) := \int_0^x \frac{1}{a(t)}\,\mathrm{d}t,$$

and let

$$I := \Phi(\mathbb{R}) \subseteq \mathbb{R}$$

As before, Φ is a smooth diffeomorphism from $\mathbb{R}$ onto I. Let $\Phi^{-1} : I \to \mathbb{R}$ denote the inverse diffeomorphism. Fix $y_0 \in I$, and define

$$W : I \to \mathbb{R}, \quad W(y) := \exp\left(\int_{y_0}^{y} b\big(\Phi^{-1}(s)\big)\,\mathrm{d}s\right)$$

Then by the Fundamental Theorem of Calculus,

$$W' = W\big(b \circ \Phi^{-1}\big)$$

Define the function

$$F := W\big(f \circ \Phi^{-1}\big) : I \to \mathbb{R}$$

Then, using the same calculation as in the first extension,

$$\begin{aligned} F' &= W'\big(f \circ \Phi^{-1}\big) + W\big((af') \circ \Phi^{-1}\big) \\ &= W\big((bf) \circ \Phi^{-1}\big) + W\big((af') \circ \Phi^{-1}\big) \\ &= W\big((Lf) \circ \Phi^{-1}\big) \end{aligned}$$

Applying the same computation to Lf gives

$$\begin{aligned} F'' &= W'\big((Lf) \circ \Phi^{-1}\big) + W\big((a(Lf)') \circ \Phi^{-1}\big) \\ &= W\big((b(Lf)) \circ \Phi^{-1}\big) + W\big((a(Lf)') \circ \Phi^{-1}\big) \\ &= W\big((L(Lf)) \circ \Phi^{-1}\big) \\ &= W\big((L^2 f) \circ \Phi^{-1}\big) \end{aligned}$$

Applying the same identity inductively gives

$$F^{(n)} = W\big((L^n f) \circ \Phi^{-1}\big)$$

on I for every $n \geq 0$.

Now define

$$G : I \to \mathbb{R}, \quad G(y) := \lim_{n \to \infty} F^{(n)}(y)$$

This limit exists because for every $y \in I$,

$$G(y) = W(y) \lim_{n \to \infty} (L^n f)\big(\Phi^{-1}(y)\big) = W(y) g\big(\Phi^{-1}(y)\big)$$

Thus

$$G = W\big(g \circ \Phi^{-1}\big)$$

Now $F \in C^\infty(I)$, and we have just shown that $\lim_{n\to\infty} F^{(n)}(y)$ exists for every $y \in I$. The proof of the original problem applies locally on any open interval, so applying it to $F : I \to \mathbb{R}$ gives

$$G' = G$$

Then

$$G' = W'(g \circ \Phi^{-1}) + W(g \circ \Phi^{-1})' = W(b \circ \Phi^{-1})(g \circ \Phi^{-1}) + W(g \circ \Phi^{-1})'$$

Since $G' = G = W(g \circ \Phi^{-1})$, we get

$$(g \circ \Phi^{-1})' + (b \circ \Phi^{-1})(g \circ \Phi^{-1}) = g \circ \Phi^{-1}$$

Now compose this identity with Φ. Since $g = (g \circ \Phi^{-1}) \circ \Phi$, the chain rule gives

$$g' = \left((g \circ \Phi^{-1})' \circ \Phi\right)\Phi' = \frac{(g \circ \Phi^{-1})' \circ \Phi}{a}$$

Thus

$$\left((g \circ \Phi^{-1})' \circ \Phi\right) = ag'$$

Also, $((b \circ \Phi^{-1}) \circ \Phi) = b$ and $((g \circ \Phi^{-1}) \circ \Phi) = g$. Therefore composing with Φ gives

$$ag' + bg = g$$

Therefore $Lg = g$.

This also gives an explicit description of the possible limits. Indeed,

$$ag' + bg = g$$

is equivalent to

$$g' = \frac{1-b}{a} g$$

Thus either $g = 0$, or for some constant C,

$$g(x) = C \exp\left(\int_{x_0}^{x} \frac{1-b(t)}{a(t)}\, \mathrm{d}t\right)$$

This recovers the previous cases: if $b = 0$, then g is a scalar multiple of $\exp\left(\int \left(\frac{1}{a}\right)\right)$, while if $b = 1$, then g is constant. $\square$

3 Degenerate leading coefficients

3.1 A simple zero: $Lf = xf'$

Theorem 3.1 **Iteration of $x\frac{\mathrm{d}}{\mathrm{d}x}$**

Let $Lf := xf'$ and let $f \in C^\infty(\mathbb{R})$. Suppose that

$$g(x) := \lim_{n \to \infty} L^n f(x)$$

exists for every $x \in \mathbb{R}$. Then f is affine. In particular, there are $a, b \in \mathbb{R}$ such that $f(x) = a + bx$ and $g(x) = bx$, so g is smooth and $Lg = g$.

Proof: This is the simplest example where the coefficient of f' vanishes, so the previous change of variables cannot be applied globally. We use the following form of the Phragmén–Lindelöf uniqueness theorem.

Lemma 3.2 **Phragmén–Lindelöf uniqueness**

Let $H : \mathbb{C} \to \mathbb{C}$ be entire. Suppose there are constants $C, A > 0$, $\tau \geq 0$, $\alpha > \tau$, and $T \in \mathbb{R}$ such that

$$|H(z)| \leq C \exp(\tau |z|)$$

for every $z \in \mathbb{C}$, and

$$|H(t)| \leq A \exp(\alpha t)$$

for every real $t \leq T$. Then $H = 0$.

The proof has three steps: conjugate L to the ordinary derivative on each half-line, control the growth of the resulting entire functions, and use their behaviour near 0 to force them to be affine-exponential.

Fix $\sigma \in \{-1, 1\}$ and define

$$I_\sigma := \begin{cases} (0, \infty) & \text{if } \sigma = 1 \\ (-\infty, 0) & \text{if } \sigma = -1 \end{cases}$$

and

$$\Phi_\sigma : I_\sigma \longrightarrow \mathbb{R}, \quad \Phi_\sigma(x) := \log(\sigma x).$$

Then Φ_σ is a smooth diffeomorphism with inverse

$$\Phi_\sigma^{-1} : \mathbb{R} \longrightarrow I_\sigma, \quad \Phi_\sigma^{-1}(y) = \sigma \exp(y).$$

Define

$$F_\sigma := f \circ \Phi_\sigma^{-1} : \mathbb{R} \longrightarrow \mathbb{R}.$$

Claim 1: For every $n \geq 0$,

$$F_\sigma^{(n)} = (L^n f) \circ \Phi_\sigma^{-1}.$$

Verification of Claim 1: Since

$$\left(\Phi_\sigma^{-1}\right)' = \Phi_\sigma^{-1},$$

the chain rule gives

$$\begin{aligned} F_\sigma{}' &= \left(f' \circ \Phi_\sigma^{-1}\right)\left(\Phi_\sigma^{-1}\right)' \\ &= \left(f' \circ \Phi_\sigma^{-1}\right)\Phi_\sigma^{-1} \\ &= (Lf) \circ \Phi_\sigma^{-1}. \end{aligned}$$

Applying the same computation to Lf gives

$$\begin{aligned} F_\sigma{}'' &= \left((Lf)' \circ \Phi_\sigma^{-1}\right)\left(\Phi_\sigma^{-1}\right)' \\ &= \left((Lf)' \circ \Phi_\sigma^{-1}\right)\Phi_\sigma^{-1} \\ &= \left(L^2 f\right) \circ \Phi_\sigma^{-1}. \end{aligned}$$

Applying the identity inductively,

$$F_\sigma^{(n)} = (L^n f) \circ \Phi_\sigma^{-1}$$

for every $n \geq 0$.

By Claim 1, Theorem 1.2 applies to F_σ, so F_σ is the restriction of an entire function, which we continue to denote by F_σ. Set

$$a := f(0), \quad b := f'(0),$$

and define the entire function

$$H_\sigma(z) := F_\sigma(z) - a - \sigma b \exp(z).$$

Claim 2: There is a constant $C_\sigma > 0$ such that

$$|H_\sigma(z)| \leq C_\sigma \exp(|z|)$$

for every $z \in \mathbb{C}$.

Verification of Claim 2: Since $\Phi_\sigma^{-1}(0) = \sigma$, we have

$$F_\sigma^{(n)}(0) = (L^n f)(\sigma).$$

Thus the sequence $\left(F_\sigma^{(n)}(0)\right)_{n=0}^{\infty}$ converges and is therefore bounded. Choose $M_\sigma > 0$ such that

$$\left|F_\sigma^{(n)}(0)\right| \leq M_\sigma$$

for every $n \geq 0$. The Taylor series of F_σ at 0 then gives, for every $z \in \mathbb{C}$,

$$|F_\sigma(z)| \leq \sum_{n=0}^{\infty} \frac{M_\sigma}{n!}|z|^n = M_\sigma \exp(|z|).$$

Since $|\exp(z)| \leq \exp(|z|)$, it follows that

$$|H_\sigma(z)| \leq (M_\sigma + |a| + |b|) \exp(|z|).$$

Thus we may take $C_\sigma = M_\sigma + |a| + |b|$.

Claim 3: There are constants $K > 0$ and $Y \in \mathbb{R}$ such that

$$|H_\sigma(y)| \leq K \exp(2y)$$

for every real $y \leq Y$.

Verification of Claim 3: By Taylor's theorem, there are constants $\delta, K > 0$ such that

$$|f(x) - a - bx| \leq Kx^2$$

whenever $|x| < \delta$. Choose $Y \in \mathbb{R}$ such that $\exp(Y) < \delta$. If $y \leq Y$, then

$$\left|\Phi_\sigma^{-1}(y)\right| = \exp(y) < \delta,$$

and hence

$$\left|F_\sigma(y) - a - b\Phi_\sigma^{-1}(y)\right| \leq K \exp(2y).$$

Since $\Phi_\sigma^{-1}(y) = \sigma \exp(y)$, this is exactly

$$|H_\sigma(y)| \leq K \exp(2y)$$

for every real $y \leq Y$.

Claims 2 and 3 verify the two growth hypotheses of Lemma 3.2 with $\tau = 1$ and $\alpha = 2$. Therefore $H_\sigma = 0$, and hence

$$F_\sigma(y) = a + b\Phi_\sigma^{-1}(y)$$

for every $y \in \mathbb{R}$. Composing this identity with Φ_σ gives

$$f(x) = a + bx$$

for every $x \in I_\sigma$.

Since $\sigma \in \{-1, 1\}$ was arbitrary, this identity holds on both half-lines. Since $f(0) = a$, we conclude that

$$f(x) = a + bx$$

for every $x \in \mathbb{R}$.

It follows that

$$Lf(x) = xf'(x) = bx$$

and

$$L(bx) = bx.$$

Thus $L^n f(x) = bx$ for every $n \geq 1$, and consequently

$$g(x) = bx.$$

In particular, g is smooth and satisfies

$$Lg = g.$$

□

3.2 A second-order zero: $Lf = x^2 f'$

Theorem 3.3 **Iteration of $x^2 \frac{\mathrm{d}}{\mathrm{d}x}$**

Let $Lf := x^2 f'$ and let $f \in C^\infty(\mathbb{R})$. Suppose that

$$g(x) := \lim_{n \to \infty} L^n f(x)$$

exists for every $x \in \mathbb{R}$. Then, for some $C \in \mathbb{R}$,

$$g(x) = \begin{cases} C \exp\left(-\frac{1}{x}\right) x > 0 \\ 0 \qquad\qquad\quad x \leq 0. \end{cases}$$

In particular, $g \in C^\infty(\mathbb{R})$ and $Lg = g$.

Proof: Define

$$\begin{aligned} \Phi_+ : (0, \infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto -\frac{1}{x} \end{aligned}$$

Note that Φ_+ is self-inverse, so $\left(\Phi_+^{-1}\right)'(x) = {\Phi_+}'(x) = \frac{1}{x^2}$ for all $x \in (0, \infty)$. Denote $F_+ = f \circ \Phi_+^{-1}$.

Lemma 3.4 **Conjugacy on the positive half-line**

For all $n \in \mathbb{N}$,

$$F_+^{(n)} = (L^n f) \circ \Phi_+^{-1}.$$

Proof: Applying the chain rule, we have

$${F_+}' = \left(f \circ \Phi_+^{-1}\right)' = \left(f' \circ \Phi_+^{-1}\right) \cdot \left(\Phi_+^{-1}\right)'.$$

Therefore, for $y \in (-\infty, 0)$, we have

$$\begin{aligned} {F_+}'(y) &= f'\left(\Phi_+^{-1}(y)\right)\left(\Phi_+^{-1}\right)'(y) \\ &= f'\left(-\frac{1}{y}\right)\left(\frac{1}{y^2}\right) \\ &= f'\left(\Phi_+^{-1}(y)\right)\left(\Phi_+^{-1}(y)\right)^2 \\ &= (Lf)\left(\Phi_+^{-1}(y)\right). \end{aligned}$$

So ${F_+}' = (Lf) \circ \Phi_+^{-1}$.

We also have

$$\begin{aligned} F_+^{(2)} &= \left((Lf) \circ \Phi_+^{-1}\right)' \\ &= \left((Lf)' \circ \Phi_+^{-1}\right)\left(\Phi_+^{-1}\right)' \end{aligned}$$

Therefore, for $y \in (-\infty, 0)$, we have

$$\begin{aligned} F_+^{(2)}(y) &= (Lf)'\left(\Phi_+^{-1}(y)\right)\left(\Phi_+^{-1}(y)\right)^2 \\ &= \left(L^2 f\right)\left(\Phi_+^{-1}(y)\right). \end{aligned}$$

So $F_+^{(2)} = \left(L^2 f\right) \circ \Phi_+^{-1}$.

Inductively,

$$F_+^{(n)} = (L^n f) \circ \Phi_+^{-1}$$

which concludes the proof. □

Note that $F_+ \in C^\infty((-\infty, 0))$. By Lemma 3.4 and the hypothesis on $L^n f$, the limit $\lim_{n\to\infty} F_+^{(n)}(y)$ exists for every $y \in (-\infty, 0)$. Thus Lemma 1.3 applies directly to F_+. If

$$G_+(y) := \lim_{n\to\infty} F_+^{(n)}(y),$$

then ${G_+}' = G_+$ on $(-\infty, 0)$, so

$$G_+(y) = C_+ \exp(y)$$

for some $C_+ \in \mathbb{R}$. Note that $G_+ = g \circ \Phi_+^{-1}$, so $g = G_+ \circ \Phi_+$. Therefore, for $x > 0$ we get

$$g(x) = C_+ \exp\left(-\frac{1}{x}\right)$$

Now, define

$$\begin{aligned} \Phi_- : (-\infty, 0) &\longrightarrow \mathbb{R} \\ x &\longmapsto -\frac{1}{x} \end{aligned}$$

and denote $F_- = f \circ \Phi_-^{-1}$.

Lemma 3.5 **Conjugacy on the negative half-line**

For all $n \in \mathbb{N}$,

$$F_{-}^{(n)} = (L^n f) \circ \Phi_{-}^{-1}.$$

Proof: The proof is analogous to that of Lemma 3.4. $\square$

Analogously, Lemma 1.3 gives

$$G_{-}(y) := \lim_{n\to\infty} F_{-}^{(n)}(y) = C_{-} \exp(y)$$

for some $C_{-} \in \mathbb{R}$.

Lemma 3.6 **Vanishing on the negative half-line**

We have $C_{-} = 0$.

Proof: First note that f is continuous at 0 because it is smooth, so

$$\lim_{y\to\infty} F_{-}(y) = \lim_{y\to\infty} f\left(-\frac{1}{y}\right) = f\left(\lim_{y\to\infty} -\frac{1}{y}\right) = f(0)$$

In particular, F_{-} is bounded for all sufficiently large y. Thus there exist $Y > 0$ and $M > 0$ such that

$$|F_{-}(y)| \leq M$$

for every $y \geq Y$.

By Lemma 1.3, F_{-} agrees on $(0, \infty)$ with the restriction of an entire function $\tilde{F}_{-}$. Therefore, for every $y_0 > 0$, its Taylor series centered at y_0 has infinite radius of convergence. Set

$$H(y) := F_{-}(y) - C_{-} \exp(y)$$

on $(0, \infty)$. Since entire functions are closed under linear combinations, the function

$$\tilde{H}(z) := \tilde{F}_{-}(z) - C_{-} \exp(z)$$

is entire. Moreover, for every $y \in (0, \infty)$,

$$\tilde{H}(y) = F_{-}(y) - C_{-} \exp(y) = H(y).$$

Thus H agrees on $(0,\infty)$ with the restriction of the entire function $\tilde{H}$. Also, for every $y > 0$,

$$H^{(n)}(y) = F_-^{(n)}(y) - C_- \exp(y) \xrightarrow[n\to\infty]{} 0$$

because $F_-^{(n)}(y) \xrightarrow[n\to\infty]{} C_- \exp(y)$.

Fix $y_0 > 0$, and define

$$a_k := H^{(k)}(y_0).$$

For $r > 0$, the Taylor expansion at y_0 gives

$$H(y_0 + r) = \sum_{k=0}^{\infty} \frac{a_k}{k!} r^k$$

and hence

$$\exp(-r) H(y_0 + r) = \sum_{k=0}^{\infty} a_k \exp(-r) \frac{r^k}{k!}$$

We claim that the right hand side tends to 0 as $r \to \infty$. Let $\varepsilon > 0$. Since $a_k \to 0$, choose K such that $|a_k| < \frac{\varepsilon}{2}$ for every $k \geq K$. Then

$$\begin{aligned} |\exp(-r) H(y_0 + r)| \leq & \sum_{k=0}^{K-1} |a_k| \exp(-r) \frac{r^k}{k!} \\ & + \frac{\varepsilon}{2} \sum_{k=K}^{\infty} \exp(-r) \frac{r^k}{k!}. \end{aligned} \tag{3.1}$$

However,

$$\sum_{k=0}^{K-1} |a_k| \exp(-r) \frac{r^k}{k!} \xrightarrow[r\to\infty]{} 0$$

because $r^k \in o(\exp(r))$ for any $k \in \mathbb{N}$. Note further that

$$\frac{\varepsilon}{2} \sum_{k=K}^{\infty} \exp(-r) \frac{r^k}{k!} \leq \frac{\varepsilon}{2} \exp(-r) \exp(r) = \frac{\varepsilon}{2}$$

because $\frac{r^k}{k!} \geq 0$ for all $r \geq 0$.

Combining these estimates with (3.1), we have

$$|\exp(-r)H(y_0 + r)| \leq \sum_{k=0}^{K-1} |a_k| \exp(-r)\frac{r^k}{k!} + \varepsilon.$$

The finite sum is less than $\frac{\varepsilon}{2}$ for all sufficiently large r. Hence

$$|\exp(-r)H(y_0 + r)| < \varepsilon$$

for all sufficiently large r. Since $\varepsilon > 0$ was arbitrary,

$$\lim_{r\to\infty} \exp(-r)H(y_0 + r) = 0$$

Now let $y = y_0 + r$. Then

$$\begin{aligned} \exp(-y)H(y) &= \exp(-y_0 - r)H(y) \\ &= \exp(-y_0) \exp(-r)H(y_0 + r). \end{aligned}$$

Since $\exp(-y_0)$ is constant and $r \to \infty$ if and only if $y \to \infty$, it follows that

$$\lim_{y\to\infty} \exp(-y)H(y) = 0.$$

Now,

$$\begin{aligned} \exp(-y)F_-(y) &= \exp(-y)(C_- \exp(y) + H(y)) \quad \text{(by definition of } H) \\ &= C_- + \exp(-y)H(y), \end{aligned}$$

so

$$\lim_{y\to\infty} \exp(-y)F_-(y) = C_-. \tag{3.2}$$

However, by the choice of Y and M, for every $y \geq Y$ we have

$$|\exp(-y)F_-(y)| \leq \exp(-y)|F_-(y)| \leq M \exp(-y).$$

Since $\lim_{y\to\infty} M \exp(-y) = 0$, the squeeze theorem gives

$$\lim_{y\to\infty} \exp(-y)F_-(y) = 0.$$

Therefore, by (3.2), we have $C_- = 0$. □

Lemma 3.7 **Smooth gluing lemma**

Let $h \in C^\infty((0, \infty))$ and suppose that

$$\lim_{x \to 0^+} h^{(n)}(x) = 0$$

for every $n \geq 0$. Then the extension

$$\tilde{h}(x) = \begin{cases} h(x) x > 0 \\ 0 \quad\quad x \leq 0 \end{cases}$$

belongs to $C^\infty(\mathbb{R})$.

Proof: See Francis, Lemma 6.2 , which proves the more general extension-by-zero principle in $\mathbb{R}^d$. □

Since $C_- = 0$, we have $G_- \equiv 0$, yet $G_- = g \circ \Phi_-^{-1}$ with $\Phi_-^{-1}(y) = -\frac{1}{y} \neq 0$ for any $y \in (0, \infty)$. Therefore, $g(x) = 0$ for $x < 0$. Also, at $x = 0$, we have $(Lh)(0) = 0$ for every smooth function h, so $g(0) = 0$. Therefore any pointwise limit has the form

$$g(x) = \begin{cases} C_+ \exp\left(-\frac{1}{x}\right), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Note that g is clearly smooth on $\mathbb{R}^*$. On $(-\infty, 0]$, every derivative of g is zero. For every $n \geq 0$, the nth derivative of the right-hand piece has the form

$$P_n\left(\frac{1}{x}\right) \exp\left(-\frac{1}{x}\right) = \frac{P_n\left(\frac{1}{x}\right)}{\exp\left(\frac{1}{x}\right)}$$

for some polynomial P_n, which tends to 0 as $x \to 0^+$ because exponential growth dominates polynomial growth. Thus, by Lemma 3.7, g is smooth at 0, and therefore $g \in C^\infty(\mathbb{R})$, as desired.

Remark 3.8 **Why C_+ need not vanish**

The constant C_+ need not be zero. On the negative half-line, $x \to 0^-$ corresponds to $y \to \infty$, where F_- is bounded and $\exp(-y) \to 0$, forcing $C_- = 0$. On the positive half-line, $x \to 0^+$ corresponds instead to $y \to -\infty$, where $\exp(-y) \to \infty$, so the same argument fails. Indeed, for any $C \in \mathbb{R}$, the smooth function

$$f_C(x) = \begin{cases} C\exp\left(-\frac{1}{x}\right), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

satisfies $Lf_C = f_C$. Hence $L^n f_C = f_C$ for every n, showing that any value $C_+ = C$ can occur.

□

4 Structural decomposition

4.1 Three important spaces

Definition 4.1 **Transient, input, and fixed-point spaces**

Let $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be a linear differential operator, and define

$$A := \left\{ h \in C^\infty(\mathbb{R}) \;\middle|\; \lim_{n\to\infty} L^n h(x) = 0 \text{ for every } x \in \mathbb{R} \right\}$$

This is the transient space. Define

$$B := \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \lim_{n\to\infty} L^n f(x) \text{ exists for every } x \in \mathbb{R} \right\}$$

This is the input space. Finally, define

$$C := \{ g \in C^\infty(\mathbb{R}) \mid Lg = g \}$$

This is the fixed-point space.

Proposition 4.2 **Transient–fixed-point decomposition**

Assume that, for every $f \in B$, the pointwise limit

$$g(x) := \lim_{n \to \infty} L^n f(x)$$

belongs to $C^\infty(\mathbb{R})$ and satisfies $Lg = g$. Then

$$B = A + C.$$

Proof: First let $f \in B$, and define

$$g(x) := \lim_{n \to \infty} L^n f(x)$$

By the hypothesis, $g \in C$. Let

$$h := f - g$$

Then $h \in C^\infty(\mathbb{R})$, and for every $x \in \mathbb{R}$,

$$L^n h(x) = L^n f(x) - L^n g(x) = L^n f(x) - g(x) \to 0$$

Thus $h \in A$, and therefore

$$f = h + g \in A + C$$

So $B \subseteq A + C$.

Conversely, let $f \in A + C$. Then there exist $h \in A$ and $g \in C$ such that

$$f = h + g$$

Since $Lg = g$, we have $L^n g = g$ for every $n \geq 0$. Hence, for every $x \in \mathbb{R}$,

$$L^n f(x) = L^n h(x) + L^n g(x) = L^n h(x) + g(x) \to g(x)$$

Therefore the limit $\lim_{n \to \infty} L^n f(x)$ exists for every $x \in \mathbb{R}$, so $f \in B$. Hence

$$A + C \subseteq B$$

Combining the two inclusions gives

$$B = A + C$$

□