# Complex Analysis

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# 1 Complex Number System

**Definition 1.1** **Complex Numbers**

The set of complex numbers $\mathbb{C}$ is $\mathbb{R}^2$, together with the operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

**Theorem 1.1**

$\mathbb{C}$ is a field. With $0 = (0, 0),\ -(x, y) = (-x, -y),\ 1 = (1, 0)$ and

$$(x, y)^{-1} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right) \text{ for } (x, y) \neq (0, 0)$$

**Notation**

If $z = (x, y)$ we call $x$ the "real part" of $z$ and write $x = \operatorname{Re}(z)$.
We call $y$ the "imaginary part" of $z$ and write $y = \operatorname{Im}(z)$.
Define $i = (0, 1)$.

**Remark**

There is an embedding

$$\begin{aligned} \mathbb{R} &\hookrightarrow \mathbb{C} \\ x &\longmapsto (x, 0) \end{aligned}$$

so we view $\mathbb{R} \subset \mathbb{C}$.

**Notation**

Under the above embedding, any $z \in \mathbb{C}$ can be written as $z = \operatorname{Re}(z) + \operatorname{Im}(z) \cdot i$
and $i^2 = -1$

**Definition 1.2** **Conjugation**

Define *conjugation* to be the map

$$\begin{aligned} \overline{\phantom{x}} : \mathbb{C} &\longrightarrow \mathbb{C} \\ z = x + yi &\longmapsto x - yi = \overline{z} \end{aligned}$$

**Note**

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$$

**Note**

If $z = x + yi \in \mathbb{C}$ it has a polar representation $(x, y) \mapsto (r, \theta)$ via $z = r\cos\theta + ir\sin\theta$ where $r = \sqrt{x^2 + y^2} \geq 0, x = r\cos\theta, y = r\sin\theta$. The angle $\theta$ is defined modulo $2\pi$ when $z \neq 0$, and is arbitrary when $z = 0$.

**Remark**

If $z_1, z_2$ have polar representations

$$z_1 : (r_1, \theta_1), \quad z_2 : (r_2, \theta_2)$$

Then

$$\begin{aligned} z_1 \cdot z_2 &= (r_1\cos\theta_1 + ir_1\sin\theta_1) \cdot (r_2\cos\theta_2 + ir_2\sin\theta_2) \\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$