Measure Theory

Contents

Problem

Consider a random (infinite) string of 0's and 1's

$$(\ell_1, \dots, \ell_n, \dots) \quad \ell_n \in \{0, 1\} \forall n \in \mathbb{N}$$

What is the probability that somewhere along the string, I find 100 consecutive zeros?

Comment

The statement of the problem is unclear. Lebesgue's machinery will clarify this

Define

$$X = \{x = (\ell_1, \dots, \ell_n, \dots) \mid \ell_n \in \{0, 1\} \forall n \in \mathbb{N}\}$$

This is, in a natural way, a compact metric space. Use the "Hamming-style" distance:
For $x = (\ell_1, \dots, \ell_n, \dots)$ and $x' = (\ell_1', \dots, \ell_n', \dots)$

$$d(x, x') = \sum_{n \in \mathbb{N}} \frac{|\ell_n - \ell_n'|}{2^n}$$

Since this is a metric space, we can consider the topology:

$$\mathcal{T} = \{G \subseteq X \mid G \text{ is open}\} \subseteq 2^X = \mathcal{P}(X)$$

Can consider

$$\mathcal{B} = \sigma\text{-Alg}(\mathcal{T}) \quad \text{(Borel sigma algebra)}$$

Then a probability measure is

$$P : \mathcal{B} \to [0, 1]$$

Problem 1.1 says:
Let

$$S = \left\{x = (\ell_1, \dots, \ell_n, \dots) \in X \;\middle|\; \exists m \in \mathbb{N} \text{ with } \ell_m = \ell_{m+1} = \dots = \ell_{m+99} = 0\right\}$$

check that $S \in \mathcal{B}$, and compute $P(S)$

1 Algebras of Sets

Fix a non-empty set X.

Definition 1.1 **Algebra of Sets**

$\mathcal{A} \subseteq 2^X$ is an *algebra of sets* if

1. $\varnothing \in \mathcal{A}$
2. If $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$
3. If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$

Proposition 1.2

Let $\mathcal{A}$ be an algebra of sets on X. Then

1. $X \in \mathcal{A}$
2. If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$
3. If $A, B \in \mathcal{A}$, then $A \setminus B \in \mathcal{A}$
4. If $\mathcal{B} \subseteq \mathcal{A}$ is finite, then $\bigcap \mathcal{B} \in \mathcal{A}$
5. If $\mathcal{B} \subseteq \mathcal{A}$ is finite, then $\bigcup \mathcal{B} \in \mathcal{A}$

Example 1.1

Obviously, 2^X and $\{\varnothing, X\}$ are algebras of sets on X

Lemma 1.3

Let $\mathfrak{A}$ be any collection of algebras of sets on X, then $\bigcap \mathfrak{A}$ is an algebra of sets on X.

Proof: First note if $\mathfrak{A} = \varnothing$, then $\bigcap \mathfrak{A} = 2^X$ which is an algebra of sets on X. We now proceed with checking the axioms of an algebra of sets on X.

1. For every $\mathcal{A} \in \mathfrak{A}$, we have $\varnothing \in \mathcal{A}$, thus $\varnothing \in \bigcap \mathfrak{A}$.
2. If $A \in \bigcap \mathfrak{A}$, then $A \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{A}$. Thus $X \setminus A \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{A}$, and thus $X \setminus A \in \bigcap \mathfrak{A}$.
3. If $A, B \in \bigcap \mathfrak{A}$, then $A, B \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{A}$. Thus $A \cap B \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{A}$, and thus $A \cap B \in \bigcap \mathfrak{A}$.

$\square$

Proposition 1.4

Let $\mathcal{B} \subseteq 2^X$ be any collection of subsets of X, then there exists a unique algebra of sets $\mathcal{A}$ on X with the following properties:

1. $\mathcal{B} \subseteq \mathcal{A}$
2. Whenever $\mathcal{C}$ is an algebra of sets on X with $\mathcal{B} \subseteq \mathcal{C}$, then we have $\mathcal{A} \subseteq \mathcal{C}$

Proof: Define $\mathfrak{A}$ to be the collection of all algebras of sets on X that contain $\mathcal{B}$. Then $\bigcap \mathfrak{A}$ is an algebra of sets by Lemma 1.3 and satisfies (1) and (2). Uniqueness follows from (2). $\square$

Definition 1.5 **Algebra of sets generated by a collection of subsets**

Let $\mathcal{B} \subseteq 2^X$ be any collection of subsets of X. We define the *algebra of sets generated by* $\mathcal{B}$ to be the unique algebra from Proposition 1.4. We notate it as $\operatorname{Alg}(\mathcal{B})$.